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REPORT ON RECENT RESEARCH ON NONLINEAR OSCILLATIONS

L. Mandelstam, N. Papalexi, A. Andronov,
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REPORT ON RECENT RESEARCH ON NONLINEAR OSCILLATIONS[1]

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ABSTRACT. A short summary is presented of works performed recently at the Institute of Physics of the University of Moscow and several of the laboratories on the problem of nonlinear oscillators. The discussion is limited to systems with one degree of freedom, in the most essential aspects.

Introduction

Since Dr. Balth. van der Pol had invited us, in his capacity as president of the Physics Section of the International Congress of the ISRU, to provide a summary of recent research on nonlinear oscillations performed at the Physics Institute of Moscow University, at the Central Leningrad Radio Laboratory, at the Electrophysics Institute of Leningrad and at Gorki University, we do this with great pleasure since a large part of our works in this important field is closely related to the basic research experiments of Mr. van der Pol. We shall only be able to discuss a restricted part of this subject. We shall, however, be able to report some general concepts which guided us during our research and some of the results obtained. /81*

Whereas up to rather recently the preponderant role in the theory of oscillations belonged to so-called linear systems ("small" oscillations of mechanical systems having a finite number of degrees of freedom, electric circuits, conventional problems of boundary conditions), at the present time, in the varied fields of pure and applied science (mechanics, acoustics, biology and, above all, since the coming of the electronic tube, radioengineering) interest is being applied more and more on nonlinear systems.

Systems presently being used by radioengineering for transmitting and receiving are essentially nonlinear and this is by no means a fortuitous

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[1] Report made to the Radiophysics Section of the General Assembly of the International Scientific Radio Union (ISRU), London, 12-18 September 1934.
* Numbers in the Margin indicate pagination in the foreign text.

circumstance. It is enough merely to examine -- let us take the most simple example -- a triode oscillator to see that an autonomous linear system, i.e. a device in which the current and voltage are governed by differential linear equations in which time does not explicitly enter[1], cannot have the properties which possess and should possess a transmitter. (We shall confine ourselves to autonomous systems in order to exclude those whose oscillations are transmitted from the outside. For, naturally, the system receives oscillations from an outside source, the question immediately arises as to how oscillations from this source are produced. In order to discuss the problem of the emission of the oscillations, systems functioning without external electromotive forces should be observed.) Indeed, the basic property of linear systems is that amplitude is not intrinsic to the system, but is completely a function of the initial conditions. Now, the distinguishing features of modern oscillating devices is that, on the contrary and independently of the initial state, there is established an oscillatory mode of operation whose amplitude and period are completely defined. This is why modern radioengineering has had to call upon physical concepts and a mathematical apparatus which were able to cope with nonlinear systems. The great diversity of phenomena which are revealed in nonlinear systems makes them extremely interesting from the purely physical side. It is also this diversity as well as their flexibility which has enabled the determination of the broad applications that these systems have received in the course of the last few years. /82

Since the study of differential nonlinear equations is much more difficult and complicated than that of linear equations, the tendency has naturally been from the beginning to "linearize" the problems, i.e. to treat the essentially nonlinear problems from a linear viewpoint. It cannot be denied that in order to clear up some aspects of the known phenomena, such a method can sometimes have its utility. However, since it is still incomplete, artificial and requires complementary ad hoc hypotheses, this method of linearization often leads to errors. One of these errors which is still encountered quite often is referred to below[2].

After the nonlinear systems had completely conquered the field, practice itself, revealing phenomena completely foreign to the linear systems, inspired a search for a mathematical apparatus which could cope with these phenomena. Very soon, publications appeared which purposely began from a nonlinear viewpoint. These are, above all, the remarkable works of van der Pol to which we will have occasion to refer several times. As has been stated, the results that they contain have a fundamental importance for the entire field in which we are involved. However, and this is natural, the first works had as their goal the production of tangible results rather than to develop a general and strict theory. Thus, for example, existence of periodic solutions was

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[1] We term as autonomous any system whose differential equations do not contain an explicit reference to time.

accepted as an assumption. Series were often used whose convergence was not unquestioned. Nevertheless, and let us emphasize this, the results obtained were good.

After using these methods and owing, in the first place, to the works of van der Pol, a great quantity of valuable results was produced and it was natural to pass on to more general points of view and to look for a mathematical apparatus which could cope with nonlinear problems. It is in this direction that a part of our works has been directed. It turned out that the mathematical apparatus which could cope with nonlinear oscillation problems had been in existance for a long time. On the one hand, it was contained in the famous works of Henri Poincare [1], [2], and, on the other hand, in the remarkable investigations of Liapounov [3]. /83

The relationship existing between the works of Poincare, carried forward by Birkhoff [4], as well as those of Liapounov, and our physical problems was pointed out by one of us [5]. Three things should be pointed out here. First of all, the qualitative theory of differential equations, developed by Poincare, has turned out to be very effective for the qualitative discussion of physical phenomena taking place in systems presently used in radioengineering. Nevertheless, neither the physicist, nor with greater reason the engineer, can be happy with a qualitative analysis. Another cycle of the works of Poincare supplied an apparatus which allowed the treatment of our problems on a quantitative basis. Finally, the works of Liapounov allow applying questions of stability to a mathematical discussion.

The first part of this report summarizes these mathematical theories and shows how we apply them to our problems. We shall almost disregard questions of stability. The second part discusses the theoretical and experimental aspects of some concrete questions. These partially involve problems where we have perfected and supplied with a strict mathematical base those results previously obtained by other authors. In addition, we shall examine resonance phenomena of the n-th degree and shall succinctly report certain experiments and theoretical considerations on the so-called parametric excitation phenomena.

We shall conclude by several observations on the role of statistics in oscillatory phenomena.

1. Geometric Presentation of the Movements of an Oscillatory System. Phase Plan

There is no question but that the mathematical methods concerned here are perceptibly more complicated and difficult than those used to study linear systems. This arises from the very nature of the physical problem which is far from being a simple one. Also, there is no doubt that the characteristic features of this apparatus will prove able to cope with nonlinear systems on a theoretical as well as practical basis. We believe that a mathematical

apparatus can only be accused of being cumbersome and overcomplicated when it leads to a result after a long succession of operations in which, each operation taken separately, is no physical interpretation. Now, this is not at all the case in the geometric apparatus connected with the name of Poincare. Here, each geometrical component possesses a direct physical sense. This is why this geometrical apparatus, although it is complicated, far from being an obstacle, simplifies the description and understanding of the physical phenomena involved. This well-founded method which consists in showing the /84
conduct of an oscillating system using a geometric figure has been used in science for quite some time. The idea is essentially as follows. In order to characterize the state of a system at N degrees of freedom it is necessary to provide 2N numbers (N coordinates and N velocities). These 2N numbers can be considered as specifying the position of a point in space with 2N dimensions. To each point of this space there corresponds a determined state (a determined "phase") of the system. This is why this space is called "extension in phase." In the case of the systems with a degree of freedom, this space has two dimensions. In the most simple cases it is a plane.

Let us take the most simple example of harmonic oscillator. Its equation as the form:

$$\ddot{x} + \omega_0^2 x = 0 \tag{1}$$

or

$$\left.\begin{array}{l} \dot{y} = -\omega_0^2 x \\ \dot{x} = y \end{array}\right\} \tag{2}$$

These equations likewise describe an electric circuit which has capacitance and self-induction, but no resistance if x represents, for example, the condenser load. We shall show the behavior of the oscillator with a plane related to rectangular x, $\dot{x}$ voltage axes (voltage-current plane). This will be the phase plane. To each new state of the system corresponds a new "figurative" point on the phase plane. To a succession of states of the system there corresponds a movement of the figurative point on the phase plane, or a phase trajectory.

Planck has familiarized physicists with the phase trajectories of the harmonic oscillator. They form a family of ellipses each enclosing the other and having a common center as origin. The equation $\dot{x} = y$ indicates that the figurative point is moving in a clockwise direction. The origin can be considered as an ellipse which has degenerated into a point.

The point x = 0, y = 0 is, in the case of system (2), a "singular point" since at this point $\frac{dy}{dx} = \frac{0}{0}$, the direction of the phase trajectories is indeterminate. According to the classification which is acceptable in mathematics, a singular point surrounded by a family of ellipses may be termed the center. It is clear that the singular points have, in the case of our subject, a special interest. For in order for there to be $\frac{dy}{dx} = \frac{0}{0}$, i.e. singular point, it is enough that $\frac{dy}{dt} = 0$, $\frac{dx}{dt} = 0$. However, if the current and voltage in a simple circuit are simultaneously equal to zero, the system is in equilibrium. All the states of equilibrium of the system under study are therefore shown by singular points of differential equations (2).

Each ellipse, each closed trajectory, represents a periodic phenomenon corresponding to certain suitably selected initial conditions. The origin represents a stable state of equilibrium in the sense that any little disturbance remains little. In the case under consideration, it is clear that the system under study describes, no matter what may be the initial conditions, a periodic movement except in the case where initial conditions correspond to the origin of the coordinates. /85

We shall produce in the same way the table of the phase plane of a damped oscillator which is governed by the equation:

$$\ddot{x} + 2h\dot{x} + \omega^2_0 x = 0 \tag{3}$$

or

$$\left.\begin{aligned} \dot{y} &= 2hy - \omega^2_0 x \\ \dot{x} &= y \end{aligned}\right\} \tag{4}$$

we shall assume first of all that $h^2 < \omega^2{}_0$.

These equations describe, for example, an oscillating circuit which has resistance, since x is the voltage at the condenser terminals. All the integral curves are spirals which are wound asymptotically with the origin of the coordinates (Figure 1). Each one of the spirals represents a damped oscillation corresponding to suitable initial conditions. Like the preceding example, the origin represents a state of equilibrium and is found to be a singular point, although of a new type. The singular points which are used as asymptotic points with a family of spirals are called the focus. In our case, the singular point is stable and is therefore a stable focus.

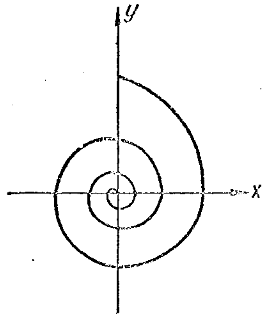

Figure 1

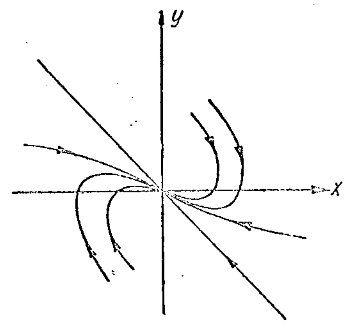

Figure 2

Giving consideration to the phase plane of a system governed by the same equations (3) or (4) but whose damping is substantial enough for $h^2 > \omega^2_0$ to be true, we shall see substituted for the spirals parabolic curves passing through the origin (Figure 2). As in the preceding, the origin represents a state of equilibrium. It is a singular point of the type called node. Integral curves reflect the aperiodic movements of the system towards the state of equilibrium. We therefore have a stable node.

In addition to the center, focus and node, there also exists an important type of singular points for us: this is the saddle (Figure 3). The saddle represents, for example, the upper equilibrium position of a pendulum. The behavior of integral curves in the vicinity of the saddle shows that the system always ends by going away. The saddle is therefore always unstable. /86

In the case of a triode oscillator whose oscillating circuit is inserted between the filament and the grid, the voltage v at the condenser terminals confirms the equation:

$$L\ddot{v} + R\dot{v} + \frac{1}{C} v = \frac{M}{C} f(v, \dot{v}) \tag{4}$$

or

$$\frac{dv}{dt} = \frac{i}{C},$$

$$\frac{di}{dt} = -\omega_0^2 v + f(v, i).$$

The form of the function f(v, i) is provided by the plate characteristic of the tube (for simplicity's sake, we disregard the grid current). The mathematical discussion of this equation with the phase plane gives the following picture . In the case of a soft mode and weak excitation (M small) the integral curves are spirals which uncoil in a closed curve. Some approach one another from the outside, coming from infinity. Others approach one another from the inside, unwinding beginning from the point of origin (Figure 4, 16). It is easy to establish the relationship between the essential lines of the geometrical figure and those of the physical system. The origin of the coordinates still represents an equilibrium state; it is an unstable focus. Even if the values of voltage and current differ very little from zero (as will be the case, for example, if the initial perturbation i_0, v_0 is produced by fluctuations), we shall witness the appearance in the circuit of oscillations whose amplitude will increase gradually. After some time, the increase will slow down, then stop and we shall see the setting up of a stationary oscillatory mode of operation which may be depicted on the phase plane by a closed curve. If the initial conditions correspond to a point located outside of the closed curve, the circuit will oscillate with a decreasing amplitude until a stationary mode of operation is set up. The /87 closed curves on which the integral curves are wound or from where they are unwound are the limiting cycles of Poincare. This mathematical concept has a very simple physical interpretation: the limiting cycles depict periodic stationary modes of operation. In the same way as the singular points, the limiting cycles can be stable (if all the neighboring integral curves come closer) or unstable (if all the neighboring integral curves draw apart). It is clear that only the stable limiting cycles represent the actual periodic movements of a physical system.

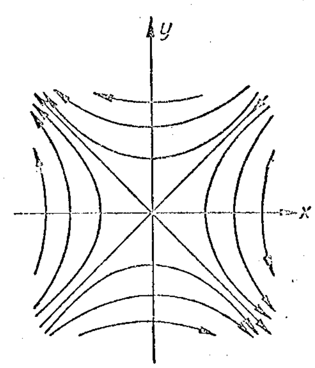

Figure 3

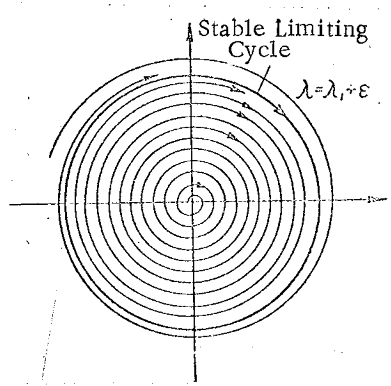

Figure 4

If a strong excitation M is given in the same case of the "soft" mode of operation, there will still exist, since the property was suitable selected, a limiting cycle with which the other integral curves will be wound from inside and from outside. Nevertheless, the behavior of the system in the vicinity of the equilibrium state will be essentially different. The integral curves will draw apart from the singular point in an aperiodic rather than an oscillatory manner and the singular point will be an unstable node. The presence of the cycle shows that there too, no matter what the initial conditions may be, there will definitely be a well-defined periodic phenomenon of "amplitude." This periodic phenomenon will be independent of the initial conditions. However, the transitory phenomenon assumes another characteristic than in the case of a minor excitation. It should be enough to study the beginning of the transitory phenomenon of a linear idealization in the case where the initial perturbation is assumed to be small. Nevertheless, the absence in the case of the unstable node of oscillatory phenomena in the linear system allows in no way a conclusion as to the absence of periodic oscillatory phenomena at a distance from the state of equilibrium at the very point at which the system cannot be considered as linear.

The case of the node enclosed in a limiting stable cycle is the most striking example of the lack of capability on the part of linear methods to decide on the existance of periodic phenomena in a self-exciting system. If this circumstance is disregarded, it is furthermore possible to commit a serious error as has been done in the case of several authors...

The singular points and the limiting cycles constitute the geometric components characterizing, to a certain degree, stationary movements in the systems. According to Poincare, the knowledge of these components is enough so as to judge the properties of all other movements. The coexistance of these components is likewise controlled by general topological laws. Also, if the properties of one of these is known, it is often possible to deduce the existance of the others. If, for example far from the point of origin, all the integral curves converge towards the origin, with the latter being an unstable focus or a node, and provided that there are no other singular points, there exists at least one limiting stable cycle. If there exist several limiting cycles enclosing one another and between which there are no singular points, there is an alternation of stable and unstable cycles. From the point of view of on-the-spot analysis, these statements are almost truisms. Nevertheless, the physical phenomena corresponding to these geometric properties are far from being trivial, and it is because it allows, by a relatively simple analysis, the production of an overview of one physical phenomenon, that the qualitative theory of Poincare is so valuable. /88

The following is a very simple example. Although without practical significance it clearly illustrates what has been said. Let us assume that the characteristic of the tube has the shape that can be seen on Figure 5. Under what conditions will the oscillator have stationary (periodic) oscillations? In order to know the behavior of the system at infinity, it is clearly possible to assume that the operational point is at the apex of the angle formed by the two rectilinear parts of the characteristic. Depending on the slope of the inclined part, two cases can be seen: either the cycle at infinity will be stable, i.e., all the integral curves will go towards infinity (if the slope exceeds a certain critical value), or else the cycle at infinity will be unstable (if the slope is less than this critical value), It can easily be seen that it is only possible to have one finite limited cycle for finite values of the slope. If the operational point is located in the horizontal part of the characteristic, the origin is a stable point of equilibrium and, consequently, owing to general topological laws, there can be no limiting cycle. Therefore, it is not possible to have oscillations. If the operational point is located on the inclined portion of the characteristic, three cases can occur. If the singular point is stable, there are no oscillations. If the slope increases, the singular point becomes unstable. If the cycle at infinity is likewise unstable, the oscillations certainly continue to exist. However, when the slope increases beyond a certain critical value, the cycle at infinity becomes stable and oscillations become again impossible.

2. Analytical Methods for Study of Nonlinear Systems

The general qualitative theory of differential equations discussed by us in part is still in the development stage. It allows analysis however incomplete in the case of two and possibly three autonomous equations (incomplete analysis in this last case) provided that the second terms are either polynomials of not too high a degree (third, fifth), or functions that can be geometrically characterized with enough simplicity. However, the radioengineer cannot be happy with the qualitative study of a problem. He requires a quantitative study which alone can be used as a base for practical calculation. On the other hand, the radioengineer accepts a quantitative theory lacking in severity and on an approximate basis provided that it takes into account satisfactorily the cases which are important for practical purposes.

From this point, the requirement is clear to prepare approximate methods of the study of nonlinear systems. These methods should, of course, take into account what these systems have in the way of specifics. One approximate quantitative method which can cope with the analysis of nonlinear systems is the one involving coefficients with slow variation, or, as we shall term it, the van der Pol method. Although this method has actually been used for quite some time in celestial mechanics, it was van der Pol who was the first to systematically apply it to problems of radioengineering. He produced a series of basic results concerning forced synchronization, "resistance," etc. [6], [7]. /89

But it was only recently that this method was supported on a mathematical basis. In addition, there still remained a certain indecisiveness in the very method of its application. The chief difficulty in this respect was obviated [8] in a way which we shall explain using equation:

(5)

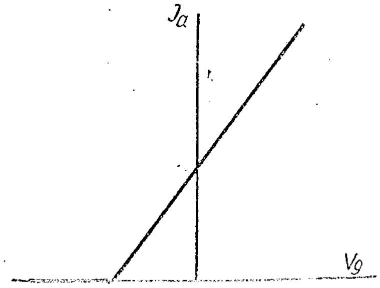

Figure 5

in which the second term is a periodic function of t of period 2π, and μ a "small parameter" on which, as we shall see, the degree of approximation will depend. It is possible to reduce to this form the equation of the regenerative receiver, etc. According to van der Pol, it should be stated:

$$x = u \cos t + v \sin t, \tag{6}$$

u and v being the functions of time t with "slow variation," i.e. whose derivatives are small with respect to u and v, and whose second derivatives are small with respect to the primary derivatives. Introducing the hypotheses involved in expressions (6) and (5), and disregarding all the terms of higher degrees as well as the harmonics, we now obtain the approximate equations of van der Pol:

$$\frac{du}{d\tau} = a_0(u, v), \qquad \frac{dv}{d\tau} = b_0(u, v)$$

in which $\tau = \mu t$ and $a_0(u, v)$, $b_0(u, v)$ are functions of u and v.

Let us take the problem from another aspect. Let us substitute for the variable x two new variables u and v defined as follows:

$$x = u \cos t + v \sin t.$$

being given that for a single variable x we substitute two variables u and v, enabling us to impose upon them the supplementary condition:

$$\dot{u} \cos t + \dot{v} \sin t = 0$$

and replacing equation (5), therefore confirmation of variable x, by the equations:

$$\left.\begin{aligned} \dot{u} &= \mu f(u \sin t - v \cos t,\ u \cos t + v \sin t) \cos t \\ \dot{v} &= \mu f(u \sin t - v \cos t,\ u \cos t + v \sin t) \sin t. \end{aligned}\right\} \tag{7}$$

Replacing the second terms by their means, we come again to the van der Pol equations or "truncated equations." By producing them in the same way, it is /90 possible to clearly state an approximation problem: it is a matter of establishing when and how much (as a function of the value μ) the solutions of the truncated equations are themselves close to those of the exact equations (7). This is a purely mathematical question which has been studied by P. Fatou [9] in a memorandum which only came to our attention after our investigations concerning the van der Pol method.

Our works, as well as the mathematical results of Fatou apply to our problems, while reporting under what conditions and in what range the truncated equations of van der Pol take into account, with a sufficient approximation, transitory phenomena. In addition, the results of Fatou allow confirming that when μ is sufficiently small, with each position of equilirium of the truncated system there corresponds a periodic solution of the exact system, and that if this position of equilibrium is stable the periodic solution is likewise so. The question relating to the mathematical base of the van der Pol method is therefore clarified. It is possible to hope that this method will likewise be justified for more complicated cases.

Let us explain in several lines those advantages of the van der Pol method. In the case of an autonomous systems with one degree of liberty, the van der Pol equations can be reduced to one alone which may be easily solved by quadrature. In the case of nonautonomous systems with one degree of freedom -- the case which has just been reported above -- the van der Pol equations are autonomous and, consequently, debatable by the methods of Poincare [29]. In the case of more complicated systems, for example those with two degrees of freedom (autonomous or subject to external effects), the van der Pol equations are systems of autonomous equations of the first degree -- two equations in the most simple cases -- which can be processed by the methods of Poincare [10][1]. The van der Pol method therefore allows replacing a system of nonlinear equations by another more simple one. It is possible to use sucessfully, in a study of the van der Pol equations, concepts of extension in phase, singular points, limiting cycles and the theory of bifurcations (section 3). We shall see below that by applying the Poincare methods to the approximate equations of van der Pol, it is possible to produce some new results which have a physical interest.

We have seen that the van der Pol method is well bases at least as far as the most simple cases are concerned. But it only provides a "zero approximation." For some questions arising in radioengineering, this would be enough. But there are other questions which require the following approximations: these chiefly concern questions of frequency adjustment, the latter only appearing in many problems as a second approximation. A theory which allows improvement of precision and calculation is therefore necessary. Such

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[1] The van der Pol equations account for three within a self-exciting system with two weakly connected circuits. This case was taken under discussion by Mayer at Gorki.

a theory unfortunately only exists in the case of purely periodic phenomena. This is the "small parameter method" to which we likewise are indebted to Poincare. This method allowed the latter to scientifically demonstrate the existance of periodic solutions of very general character for the three-body problem. In substance, this method consists of the following. Let us assume that when the parameter $\mu = 0$, our system accepts certain periodic movements. A search is then made for the movement existing when μ is not equal to 0 in the form of an ordinate series according to the powers of μ, in which the zero approximation is one of the solutions corresponding to $\mu = 0$. If, μ being zero, the system concedes a family of periodic solutions, it includes a discontinuous system of periodic solutions close ot those existing when $\mu \neq 0$ and which should be determined. This method is especially convenient when with zero approximation this system is linear and conservative[1]. /91

We have applied this method to a whole series of self-exciting problems [12], [13], [14], [15].

3. Variation of One Parameter. Stability As a Major One

In order to study the important subject of transformations undergone by the phase plane in the case of variation of one parameter, we should, once again, consult Poincare. Poincare was led, by his famous theory of the equilibrium of liquid masses in rotation, to state and brilliantly solve questions relative to the development of equilibrium states of a conservative system in the case of the variation of one parameter. The concept created by Poincare concerning the bifurcate value of the parameter can be generalized and applied to the problems in which we are involved. One value of the parameter $\lambda = \lambda_o$ is called ordinary if there exists a finite quantity ε ($\varepsilon > 0$) such as for $|\lambda - \lambda_o| < \varepsilon$. The curves integral with the phase plane have the same qualitative appearance and are bifurcate in the contrary case.

In the general case, the theory of that development undergone by the qualitative appearance of the phase plan in the case of variation of the parameter is quite complicated and insufficiently perfected. However, in the case of the approximately sinusoidal oscillations, the theory is simplified to the extreme and returns to the theory of Poincare relative to the equilibrium states of a conservative system. It is enough to replace the coordinates of the states of equilibrium by the squares of the amplitudes of stationary movements (limiting cycles and singular points) [16]. Without explaining the Poincare theory, we shall provide an example of its application.

Let us take the two chief types of excitation, the "soft" excitation and the "abrupt" excitation. We shall concern ourselves with the emitter of Figure 6 and shall select as a parameter the coefficient of mutual induction. /92

[1] Pontriaguine has provided a general method in the case in which, with zero approximation, the system is Hamiltonian.

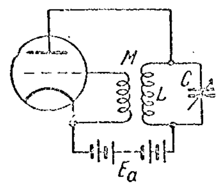

Figure 6

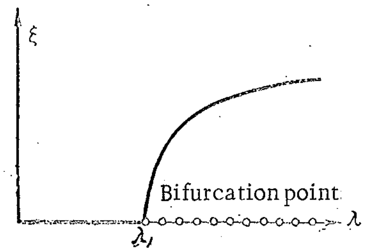

Figure 7

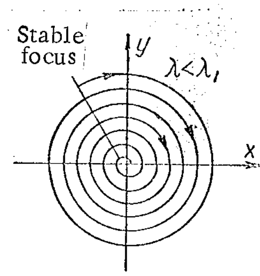

Figure 8

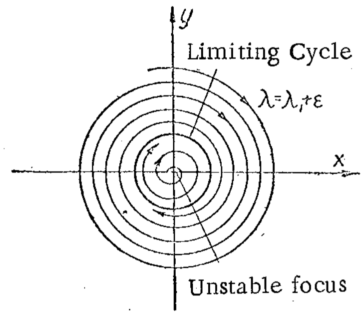

Figure 9

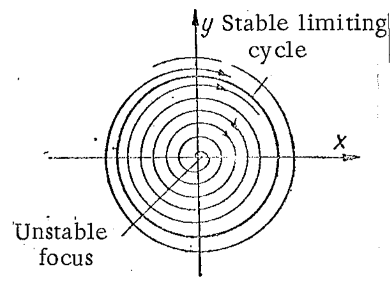

Figure 10

Soft excitation. Let $\lambda = \lambda_1$ be the value of this parameter corresponding to the excitation (Figure 7). In the case of $\lambda < \lambda_1$, the only stable stationary state is the state of equilibrium depicted by one focus (Figure 8). No matter what may be the initial position of the figurative point, at the end of a certain length of time it will be found in the proximity of this focus. $\lambda = \lambda_1$ is a bifurcative value of the parameter: the focus loses its stability at the same time as it generates a small stable limiting cycle (Figure 9), on which the figurative point begins to rotate. Passing to the language of physics, we shall say that the oscillator has been excited. λ increasing, the radius of the limiting cycle becomes larger (Figure 10), and with λ decreasing, all the phenomena are reproduced in the reverse direction: the limiting cycle is reduced to one point, and the oscillations cease. On the physical diagram I^2, (λ being the amplitude of the current), we obtain a "soft" transition from the state of equilibrium to the periodic movement and vice versa: the amplitude of the oscillations varies in a continuous manner (Figure 11).

Abrupt excitation. If, in the case of small values of λ, the system is found to be in the proximity of the state of equilibrium, it remains there

until λ assumes the value $\lambda = \lambda_1$ (Figures 12, 13, 14, 15). The creation at /93
the instant in which $\lambda = \lambda_o$ of two twin limiting cycles -- one stable and the other unstable -- does not disturb our figurative point since it leaves intact the stability of the equilibrium state. In the range $\lambda_o\lambda_1$, the unstable cycle becomes smaller, then when $\lambda = \lambda_1$, disappears and corrupts, so to say, the singular point by its instability. At this instant, the figurative point rejoins, following the integral curves, the stable limiting cycle whose amplitude has gradually increased from the instant at which $\lambda = \lambda_o$ (Figure 16). Causing the parameter to vary in reverse, we observe that on "return," the oscillator takes another path than when "going." Indeed, the figurative point will stay with the limiting cycle until the instant at which $\lambda = \lambda_o$. At this instant, the two cycles are merged, impelling the figurative point towards the state of equilibrium. The fact that when $\lambda = \lambda_1$, the latter becomes stable, produces no effect on the movement of the figurative point, since at the instant in which $\lambda = \lambda_1$ the characteristic of the cycle used as its path does not change.

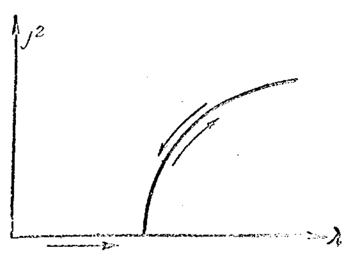

Figure 11

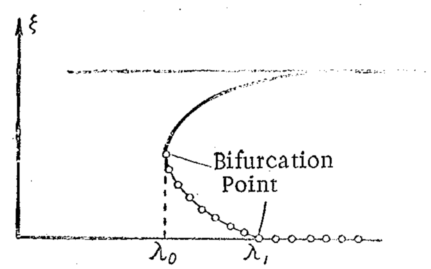

Figure 12

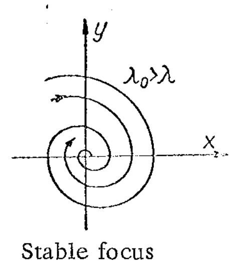

Figure 13

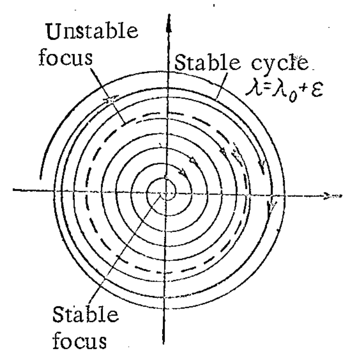

Figure 14

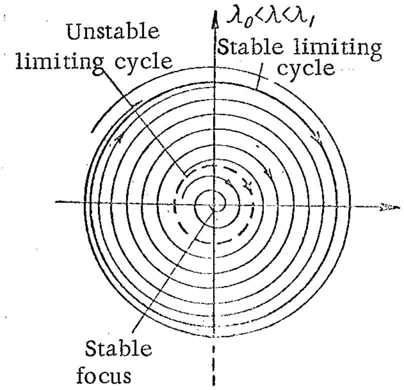

Figure 15

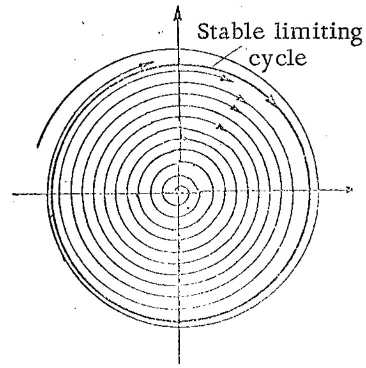

Figure 16

Diagram I^2,λ (Figure 17) emphasizes a discontinuous ("abrupt") variation of the amplitude, a variation which, owing to its irreversibility, recalls hysteresis phenomena. /94

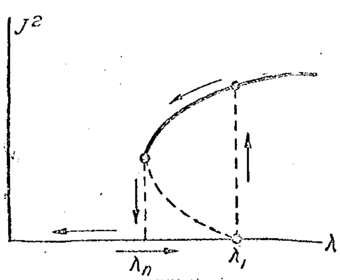

Figure 17

This phenomenon of abrupt excitation, quite interesting for the radioengineer, finds in the language of singular points, limiting cycles and bifurcative values of the parameter, its natural and proper interpretation. For this reason, it can be immediately seen that, when $\lambda_0 < \lambda < \lambda_1$ the figurative point can be "cast" from one stationary mode of operation into another by a sufficiently strong impulse. This is a claim which we suspect can be demonstrated intelligibly by semilinear theory.

We should like to discuss at this point a concept which would have no significance in the case of a conservative system, but which, in the case of a nonconservative system, has a great advantage. Let a path be stable. We can delimit, on the phase plane, a region containing the initial positions of the figurative point from which the latter, ultimately (when $t \to +\infty$) rejoins this path. This region is called the "region of stability on a large scale" or "region of attraction" of the stationary movement under consideration. Figures 18 and 19 represent two examples.

The phase plane of Figure 18 has three singular points: two stable nodes and one saddle. The range of attraction of node A is the right semiplane, whereas that of node B is the left semiplane.

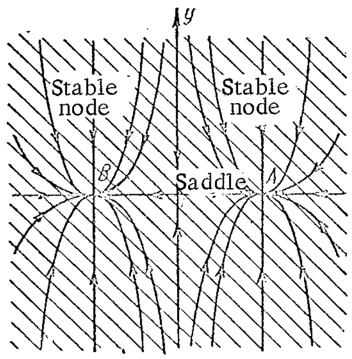

Figure 18

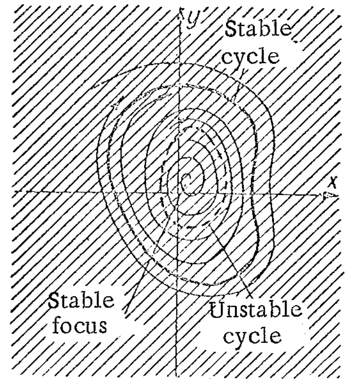

Figure 19

Let us now consider (Figure 19) a stable singular point, surrounded by an unstable cycle, contained in turn within a stable cycle. The range of stability on a large scale of the position of equilibrium is in this case the portion of the plane contained within the unstable cycle. The remainder of the plane makes up the range of attraction of the stable cycle..

Without providing more complicated examples, let us point out that the sharing of the extension in phase into fields of attraction comes up against, even in the case of one degree of freedom, some obstacles. It appears that considerations of probability must be even taken into consideration here. /95

Let us note, in conclusion, that the concept of stability on a large scale provides the unstable limiting cycles and the separating curves their true physical interpretation. These are the boundary movements similar to what is called, in geography, the tidal line: according to which the figurative point is placed, at the initial instant, on one side or the other of these components, and takes its direction towards differing destinations.

4. Autonomous Systems

The problem of autoexcitations is located in the first rank of nonlinear multiple problems, i.e. those oscillations created by the oscillating system itself without participation of any external forces varying with time and at the expense of a constant source of energy (for example, a storage battery). The qualitative questions and, partially, the quantitative questions which concern directly the study of an autonomous nonlinear system, can be solved by methods which we have described. We have already shown the appearance which provides the use of these methods to the problems relating to soft and abrupt excitation, and to transitory phenomena. Quantitative methods of approximation allow, as we have stated, finding the amplitude of autoexcitations and frequency correction in the case of practically sinusoidal oscillations. Let us comment that, in these problems, at least insofar as it only concerns zero approximation, the methods pointed out only place previously known results solidly upon a mathematical base. In this way, in certain cases, it is possible to precisely demonstrate the existence of periodic solutions and establish their stability. We estimate that these demonstrations of existence have a high degree of usefulness and the following is why.

When we set up as a differential equation any problem of physics, we are always forced to simplify it. We do not write the equation of the problem which is provided us, but that of a simplified and idealized problem. Now, can we have the certainty that we have not disregarded any of the essential features of the real problem? The situation changes if we have demonstrated the existance of a periodic solution which has an existance confirmed by experience. This demonstration is an argument to maintain that we have not omitted essential features, considering as the latter those which allow production of a stationary oscillation. But these are only indirect data. The same will not be true if it can be demonstrated that the differential equations do not have a periodic solution, whereas the system which they claim to describe does possess such solutions. We then have the certainty of having disregarded some essential feature, and we attempt to recapture it. Practical experience recognizes cases in which investigation for demonstrations of existence has suggested the means of making up for omissions of this type and placing further discussion on the proper route. /96

The following is one of the most elementary examples. Everyone knows how many manuals -- chiefly among the older ones -- describe the theory of the bell or electric switch. The armature, in the state of equilibrium, closes thecircuit of the electromagnet. When a battery is placed in the circuit, the electromagnet attracts the armature, the current is interrupted, the magnet loses the force of attraction, the spring replaces the armature into its original position and, as the expression goes -- among so many others, the game continues. If this reasoning is translated into differential equations, it is possible to show very easily that they do not permit the game to continue and that they allow only a periodic solution. Some essential point hastherefore been disregarded. In reality, the theory of switches is less simple than it appears at first glance. Self-induction is essentially necessary for oscillations to be possible at all. Mr. Leontovitch [17] has successfully discussed the problem of the switch and has clearly shown that not only is self-induction necessary for the existence of the phenomenon, but that it becomes a factor in the period of oscillations, this period differing from that of the tuning fork or armature. Other examples could be quoted showing the actual utility of investigations devoted to questions of existence.

Radioengineering often reveals conditions in which oscillations are practically sinusoidal[1]. However, during these last two years, and to a great extent owing again to the works of van der Pol, interest has grown for systems carrying out oscillations which greatly differ from the sinusoidal shape and may be termed oscillations "of relaxation." The characteristic of these oscillations is essentially a function of the resistance or parameters equivalent to it. This signifies that in the equation of the type:

$$L\ddot{q} + \frac{q}{C} = f(q, \dot{q}) - R\dot{q} = \varphi(q, \dot{q}) \tag{8}$$

which describes the autoexcitation system. The function $\phi(q, \mathring{q})$ is not limited to small values as in the case of approximately sinusoidal oscillations, but, as van der Pol has pointed out, may take on considerable values. Since it is given that concerned here are periodic solutions, this case is entirely within the purview of the qualitative theory of Poincare. The singular unstable point is here a node, and the periodic solution corresponds to a limiting cycle.

According to the most simple hypothesis, when the characteristic of the tube can be assimilated to a cubic parabola, the equation can be described in the form:

$$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0 \tag{9}$$

where, assuming $\mathring{v} = y$, in the form: /97

$$\frac{dy}{dv} - \varepsilon(1 - v^2) + \frac{v}{y} = 0. \tag{10}$$

if $\varepsilon \ll 1$ we have oscillations which are almost sinusoidal ("Thomsonian" oscillations). However, the qualitative theory is applied to the general case no matter what the value of ε may be. When $\varepsilon \ll 1$, the singular point is a focus and, when $\varepsilon \gg 1$, it is a node. In both cases there is a limiting cycle.

Nevertheless, it is possible to proceed, in quantitative studies of oscillations of relaxation, in another way. It is possible to idealize the problem, assuming that L = 0, and to replace the equation of the second order (9), by the equation of the first order:

---

[1] One very simple mechanical system allows production of autoexcitations which are almost sinusoidal. This is the Froud pendulum which has been studied by Strelkov [18].

$$\dot{v} = \frac{v}{\varepsilon(1 - v^2)} \tag{11}$$

which is easily integrated. This equation clearly does not allow a periodic solution. At the expiration of a finite time, the velocity (or the electric current, or its derivative), represented by v, becomes infinite. After having idealized the problem in this way, it is required, in order to take into account in an approximative manner the physical phenomenon, to introduce a new condition consisting, in our case, in that at a certain instant the current undergoes a discontinuity, whereas the voltage at the condenser terminals remains constant. This assumption, or "condition of discontinuity" is physically justified by the fact that the energy cannot vary discontinuously. It is possible to provide it with another form by explicitly requiring continuity of variation of energy. This "discontinuous" theory, together with the condition of discontinuity, allows revealing "discontinuous" periodic movements and finding its "amplitude" as well as its period.

This manner of processing oscillations of relaxation, applicable to electric and mechanical systems, is analogous, without being identical, to methods employed in mechanics to analyze elastic shocks. It is granted that, at the moment of shock, the velocity changes discontinuously. The preservation of energy and momentum allows reduction of velocities after the shock from those existing beforehand. This method excludes in principle the possibility of studying what is produced during the extremely short duration of the shock. The results which it gives are often sufficient since the shock reaches its term quite rapidly. However, if we wish to follow the phenomenon of shock itself, the problem becomes extremely complicated. It is enough to merely recall the investigations of Hertz. Likewise, in our theory of oscillations of relaxation, we can simplify the problem on a mathematical basis by idealizing it. In return, we do not see how the system can "leap" from one state to the other.

We have applied this method to the study of electric systems with one degreeof freedom in which the self-induction plays a secondary role [19], as well as to mechanical autoexcitation systems with small mass and high degree of friction [20].

It should be noted that, by taking into account in this case the "parasitic" self-induction, nothing is obtained which is physically of interest. /98 It remains the parasitic capacitance of conductors, etc. Now, it is impossible to take into consideration all the parasitic parameters. Our idealization has the advantage in that it allows processing relaxation systems which are relatively complicated, such as the multivibrator of Abraham-Bloch, a system with two degrees of freedom. This system had already been studied by van der Pol [21], but with one essential restriction: it assumed the phenomenon to be completely symmetrical, and rejected the consideration that there were transitory phenomena following an initial asymmetrical state. This restriction allowed him to produce an equation of the second order.

Nevertheless, in the general case, he would have obtained two equations of the second order, and this would have greatly complicated the problem. Our idealization provides, for the general case, two equations of the first order which can easily be studied by the methods described above [22]. We have thus been able to study not only the stationary mode of operation (calculate the amplitude and period), but also the transitory asymmetrical phenomena in the multivibrator of Abraham-Bloch. The results were experimentally confirmed.

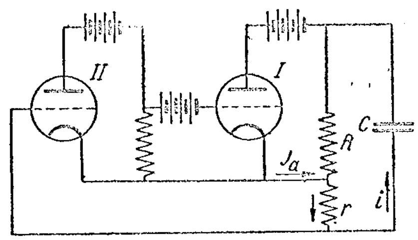

Figure 20

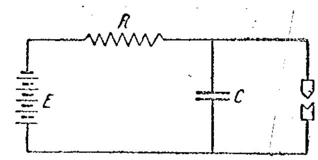

Figure 21

The discontinuous theory is applicable not only in the case in which one of the parameters is small. Even in a circuit with capacitance and self-induction, there are cases which can occur where, in certain regions, the velocity becomes so great that it is possible to replace the very swift variation of state of the system by a discontinuity and to determine the final result of the movement in this region by using the "condition of discontinuity." Removing in this way those regions where quantitative research is most difficult, we can solve with a precision satisfactory for all practical purposes a whole series of questions which involve oscillations whose form is essentially nonsinusoidal. It is possible to apply this method, for example, to the installation described in Figure 20.

The parasitic parameters again give rise to the following observation. By studying an oscillatory system, we always idealize it, disregarding certain parameters (for example, the self-induction of a condenser or the capacitance of a coil). Nevertheless, we often obtain good results. The cause for this rests in the rather extensive properties of the mathematical apparatus concerning which we shall not go into detail. However, there are cases where, by disregarding certain parameters, no matter how small they are, we change qualitatively the table of phenomena. Let us consider the system shown in Figure 21. The circuit is formed by battery E, the resistance R, the capacit- /99
ance C, and an electric arc. By using the characteristic of the arc we produce, using the customary graphic method, three stationary values of current. Analyzing by well known methods the stability of these values, we find that two of them, A and B, are stable (Figure 22). However, if we introduce a self-induction L (Figure 23), arbitrarily small, the state of equilibrium becomes unstable. Therefore, in reality it does not exist.

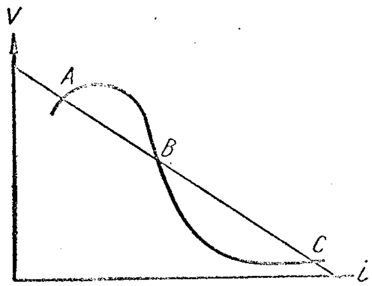

Figure 22

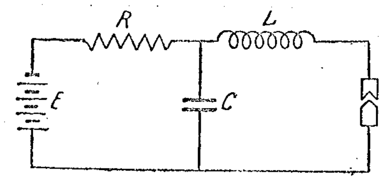

Figure 23

It is easy to see how cases of this kind can occur. If the order of the differential equation from which we draw conclusions as to the stability or instability (in the case of the equilibrium this will be an equation of the n-th linear order with constant coefficients) does not rise owing to the introduction of the parasitic parameter, the latter, if it is sufficiently small, is not able to change anything. If, on the other hand, appearing in the equations, this parameter increases their order, it can render unstable equilibrium states considered stable by disregarding it. The physical sense of this finding is clear. In composing a system of equations of the n-th order, we only assume initial conditions forming a system with n dimensions. And when, by taking into account the parasitic parameter, we raise the order of the differential equations, we concede through this a greater diversity of initial conditions. It is then possible that among the initial states newly conceded the conditions are right for the system to draw apart from the state of equilibrium. A certain caution is therefore necessary in idealization.

Some words on the self-exciting systems with distributed parameters which play an important role in radioengineering and in the mechanics (emitters containing antennas or wires of Lecher, tubes with the very high frequency waves of Grechova whose grid makes up an oscillating system, telegraphic wires emitting a sound owing to the effect of wind, vibrating airplane wings, bowed musical instruments, organ pipes, etc.). There still does not exist a precise mathematical theory for these phenomena. Nevertheless, it is possible without speaking to strictly to constitute rather easily the theory of some of them by analogy with the one, produced under rather precise conditions, existing for the system having a finite number of degrees of freedom. This theory allows the calculation of amplitudes, the solution of questions of stability, etc. [23]. However, since this theory does not have a strict basis, it is necessary to use its results with care. It takes into account the most characteristic phenomena occurring in distributed self-exciting systems. It anticipates that oscillatory modes of operation from different frequencies can be established under the same operating conditions (the same is true in the self-exciting systems having a finite number of degrees of freedom). The production of such and such an operating mode is a function of /100 the initial conditions, or the history of the system. These phenomena have been produced and studied experimentally by Bendrikov and Brailo at Moscow, as well as by Gaponov at Gorki. It is possible to cause the disappearance of a wave in a wire autoexciter of Lecher, by touching them with a finger. The system then begins to oscillate on another wavelength. When the finger is taken away, the system under certain conditions is not returned to the original wave, but keeps the new wavelength. This phenomenon can likewise be produced by capturing the energy using a resonant circuit. Similar phenomena occur apparently in bowed musical instruments. Strelkov has performed similar experiments with a string vibrating under the effect of a jet of water or air. These quite simple experiments allow the observation of phenomena which are characteristic for distributed self-exciting systems.

Given that under the same operating conditions there can occur different oscillating modes of operation, if the question may be asked as to which of them is produced when the system is triggered. This question is often within the competence of the theory of probabilities. Theoretically, it has not yet been solved for distributed systems. The statistical phenomena are easy to observe experimentally in the Lecher wire oscillator. They likewise take place in organ pipes.

5. Effect of an External Force on a Self-Exciting System

One characteristic property of self-exciting systems, quite important for the entire range of radioengineering, is the appearance of phenomena of forced or automatic synchronization or frequency drive. This phenomenon, already noted by Huygens in clocks hanging from a same wall, was observed for the first time in radioengineering by G. Moeller [24] and Vincent [25]. It occasioned many experimental and theoretical investigations among which special mention should be made of those of van der Pol [6]. In its most simple aspect, this phenomenon consists, as is well known, in the following. Causing to act on a self-exciting system of frequency w_o an external force of frequency w, there is observed, when the "out of tune" $w - w_o$ is sufficiently small, that there are no beats as would be the case in a nondamped linear system. The system is automatically synchronized on the frequency of the external force. A similar phenomenon occurs in a system subjected to the effect of a force which is not periodic, but only quasi-periodic (which can be depicted by a sum of terms of incommensurable frequencies). In this case, the drive occurs when the frequency of the system is close to one of the frequencies of the force applied or one of the combining frequencies. The /101
drive likewise occurs, as observed by van der Pol and van der Mark [26] in the realization systems and Koga [27] in the ordinary oscillators when one of the frequencies of the external force is close to a multiple of the frequency of the system. When the out-of-tune condition exceeds a certain value which can be termed a limit of drive, the appearance of beats may be ascertained. When, since the force is sinusoidal, the misalignment or out-of-tune condition exceeds by far the drive limit, it can be broadly stated that there exists in

the system two oscillations, one responding to the frequency of the external force, and the other proper to the system. Nevertheless, if the misalignment only slightly exceeds the drive limit, this last frequency is shifted towards the frequency impressed from the outside and, as we shall see, the whole phenomenon becomes complicated.

The theoretical study of the drive phenomenon consists in searching for stationary solutions of the differential equation:

$$\ddot{y}+\omega^2_0 y=\mu f(y,\ \dot{y})+\Sigma A_s \cos\ (\omega_s t+\delta_s). \tag{12}$$

In the most simple case in which the electromotive force acts upon a system in a soft mode of operation and with almost sinusoidal oscillations, the "curves of amplitude" have, as is known, the appearance shown by Figure 24. The portions 4-7-12, 5-8-11, etc. belong to the synchronization mode. The parts 1-4-12-15 belong to the beat mode. Theoretically, these curves have a symmetrical appearance, but experience shows most often asymmetrical curves (for example, those of Figure 25). This deformation is probably owing to the presence of a grid current. This is what appears to be confirmed by the recent results of Bakoulov (Moscow).

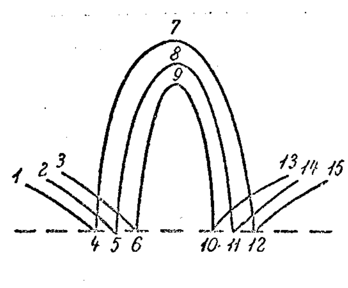

Figure 24

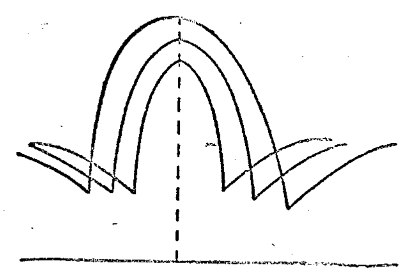

Figure 25

In his classic study, van der Pol had studied a self-exciting system with soft mode, assimilating the characteristic of the tube to a cubic parabola, and was able to detail the principal characteristics of the phenomena of synchronization and shift of frequency. Nevertheless, there are still several questions remaining. Thus, there still remains to be explained whether there exists, in phenomena of synchronization as assumed by Ollendorff [28], a "threshold" for the amplitude of the applied emf.

It was possible to show, resorting to the "truncated" equations of van der Pol set up for this problem that in the topographical analysis of Poincare, the threshold did not exist [29]. The van der Pol equations follow: /102

$$\left.\begin{aligned} \frac{dx}{d\tau} &= -ay + x(1-r^2) \\ \frac{dy}{d\tau} &= A + ax + y(1-r^2) \end{aligned}\right\} \tag{13}$$

in which $a = \frac{2(w_0 - w)}{\alpha}$ being the misalignment, w_0 being the frequency of the self-excitations, $A = -\frac{Bw_0}{\alpha a_0}$, B being the amplitude of the force used, α being a constant depending on the two parameters, a_0 being the amplitude of the self-excitations, $\tau = \frac{at}{2}$ and finally t representing time. This has also been experimentally proven [30]. It was thus easy to show quantitatively that in the case of a weak signal the relative width of the synchronization band $\frac{w_0 - w}{w}$ is provided by the ratio of the amplitude of the signal to that of the self-excitations. This is what has permitted the application to weak /103 signals of the method of measurement of intensity of the field by the width of the synchronization band as suggested by Appleton [31], [32]. A similar method, based on the phenomenon of synchronization of acoustic self-exciters, has been used to measure the intensity of sound [33], [34]. The phenomena of acoustic drive leads to an interesting problem that we are getting ready to study, which is that of the automatic synchronization of woodwind and bowed orchestra instruments.

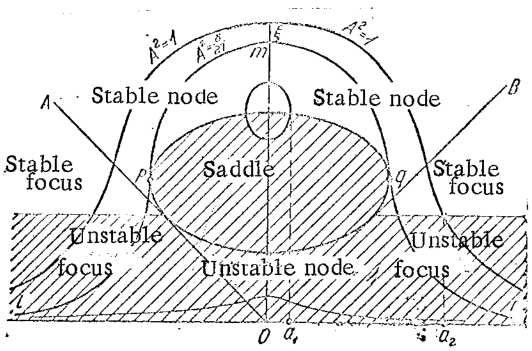

Figure 26

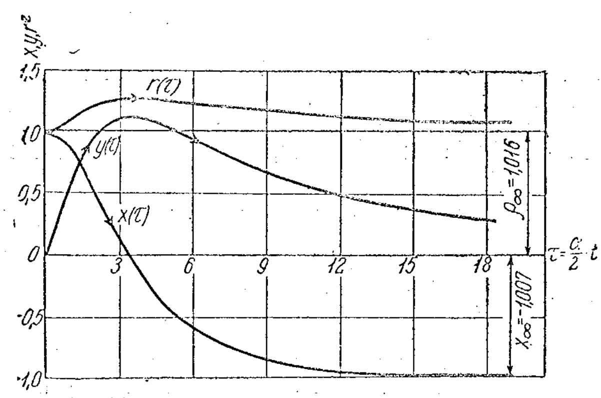

Figure 27. The Integral Curves of Differential Equations

$$\frac{dx}{d\tau} = -y + x(1-r^2)$$

$$\frac{dy}{d\tau} = 0{,}303 + 0{,}3x + y(1-r^2)$$

with initial conditions;

$$\tau = 0, \quad x_0 = 1, \quad y_0 = 0$$

$$A < \sqrt{\frac{8}{27}}$$

Let us return to the analysis of equations (13). If the square of amplitude A^2 is plotted on the axis of the ordinates and detuning a is plotted on the axis of the abscissas, Figure 26 is produced. The resonance curves are those of van der Pol, but our figure shows the fields corresponding to the various types of transitory phenomena. It is possible to take an interest in what is happening when the oscillator becomes active and an electromotive force is applied to it. It is also possible to observe what occurs when the oscillator is triggered after having applied the electromotive force. Let us examine the first case which is physically the most interesting. The theoretical study of transitory phenomena consists in discussing the nonstationary solutions of the equations concerning amplitudes (13), i.e. in following the variation of the amplitudes which are components x and y as a function of time. These results are summarized by the diagram of Figure 26. In order to know how the stationary oscillations are set up and with the unbalance and amplitude of the electromotive force applied being given, it is necessary to take the resonance curve corresponding to this emf and, on the latter, the point corresponding to the given unbalance. If this point is located in the field of the stable node, the establishment of the mode is accomplished aperiodically. The coefficients with slow variation in the van der Pol solution tend aperiodically toward constant values. If this point is in the field of stable focus, the phenomenon is oscillating. Finally, if the point falls within the range of instability, there are no stable periodic solutions. These results have been experimentally confirmed in the works of Riazine [35] who made oscillographs of various types of transitory phenomena. He calculated the solutions of equation (13) by methods of numerical integration and confirmed the theoretical results by low-frequency oscillograms. The calculated curves of aperiodic transition are shown in Figure 27, those of oscillating transition in Figure 28, and the respective oscillograms in Figures 29 and 30. Before the application of the signal there appears a pure sinusoid that the oscillograms do not show.

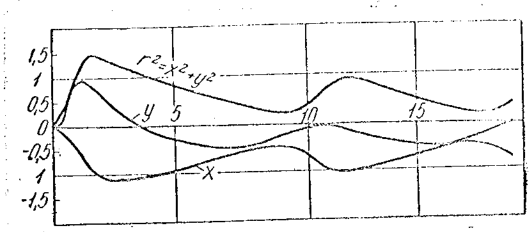

Figure 28

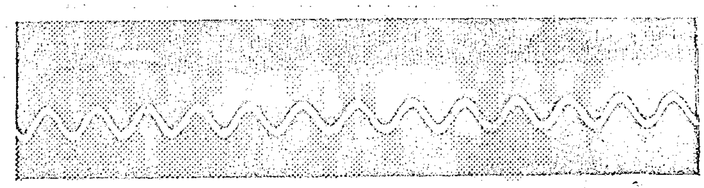

Figure 29

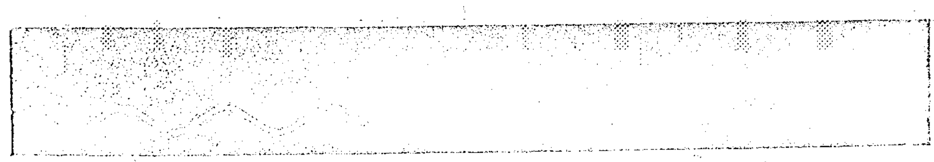

Figure 30

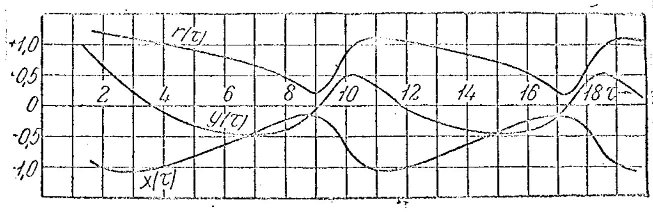

Figure 31

The same method of numerical integration was used to study what occurs to a minor degree outside of the synchronization band. In this case, the nonstationary solution of equation (13) does not tend, on the x,y plane, towards a singular point as in the field of drive, but is coiled around a limiting cycle. The curves showing current as a function of time and limiting cycle are depicted in Figures 31 and 32. The theoretical results agree closely with experimental findings (Figure 33). The theoretical curves and oscillograms agree to show that, with respect to the beats, the amplitude increases at a perceptibly greater rate than it decreases.

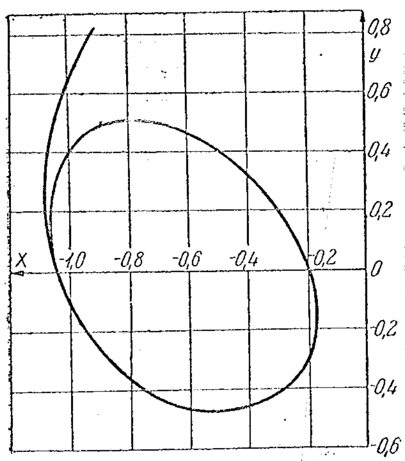

Figure 32

In order to set up the spectral composition of the beats, Riazine made the harmonic analysis of the curves of Figure 31. The spectrum obtained (Figure 34) shows that in the vicinity of the synchronization region, the application of an electromotive force causes the appearance in the self-exciter of a spectrum which has equidistant combinative frequencies clearly delineated. /105
Oscillograms were produced for the beats in the vicinity of the synchronization bands for ratios of frequency characteristic for the frequency applied which were approximately equal to 1:2, 1:3, 1:4 and 1:5 (Figures 35, 36, 37, 38). It becomes obvious from these oscillograms that in the vicinity of the limit of forced synchronization we have, as in the case of ratio 1:1, a pulsation of amplitude. The envelope of beats always shows practically the same aspect: rapid increase, then slow decrease of amplitude. Let us emphasize again that in the vicinity of the synchronization band, the oscillations cannot be expressed by the sum of two sinusoidal terms as has been done up until now. There are at least three oscillations of amplitude which are almost equal.

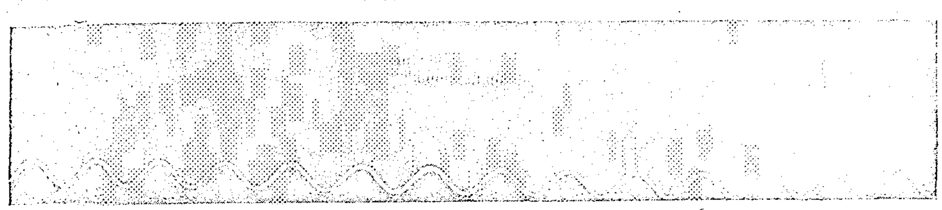

Figure 33

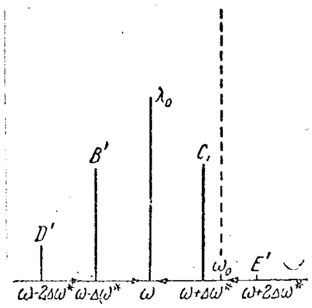

Figure 34

The study likewise was concerned with the phenomena which occur in the case of "abrupt" modes (H. Sekerska [36]). The so-called field of potential self-excitations has, in this case, a special interest. In this field, the resonance curves end in a peak, recalling those that we produced in the resonance phenomena of the n-th class (section 6). In addition, phenomena were observed there which we have termed asynchronic excitation (section 6).

There still exist phenomena of synchronization in the combinative frequencies given by several electromotive forces. We shall give as an example here a special case which we observed some time ago: the synchronization "in the middle." The oscillator is tuned approximately to the frequency $\frac{w_1 + w_2}{2}$: w_1, w_2 being the frequencies of the two electromotive forces. Synchronization phenomena are then clearly observed, and especially so when there exists a simple relationship between the frequency of the oscillator and those of the electromotive forces. It is understood that this phenomenon is an important one in the case of reception without carrier wave. In reality, when the carrier frequency is produced on the transmission site it is for practical purposes quite difficult to arrive exactly in the middle of the side frequencies, this being, moreover, absolutely necessary. The phenomenon which has just been described lends its assistance: the frequency of the oscillator is automatically located in the middle of the side frequencies. For this purpose, it is enough for the oscillator to be only approximately tuned. Some similar phenomena occur when the oscillator is tuned on the frequency $\frac{w_1 + w_2}{4}$, on the other combinative /106
frequencies and their submultiples. Theoretically, the phenomenon of "in-the-middle" synchronization has been studied by Goldstein and Petrossian[1].

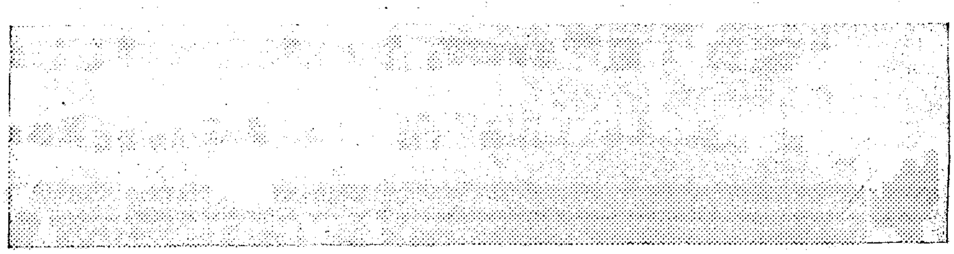

Figure 35

[1] The publication is in preparation.

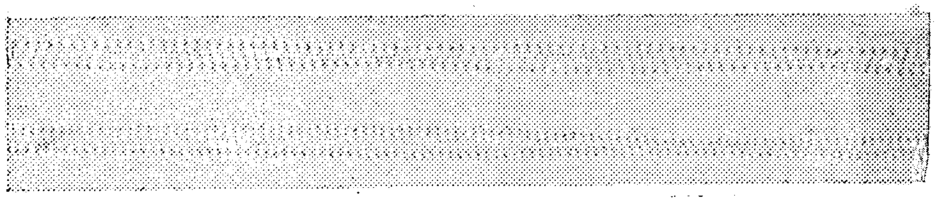

Figure 36

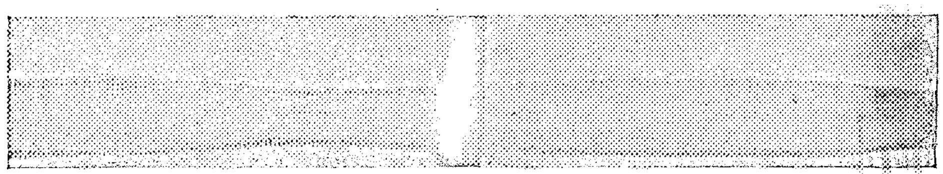

Figure 37

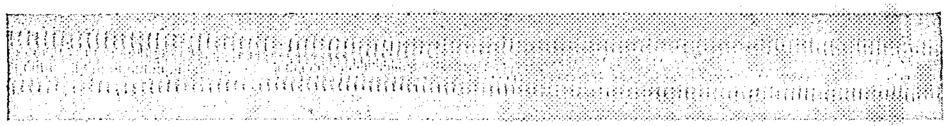

Figure 38

6. Resonance Phenomena of the n-th Class /107

We shall devote a special paragraph to the phenomena which can be termed resonance of the n-th class. The mathematical theory of these phenomena is based on the results provided generally, without any relationships to the physical applications concerning us, in the well known works of Poincare [2]. Poincare shows that in nonlinear systems there can exist periodic oscillations whose period is a multiple of that of the force applied ("periodic solutions of the second class"). It is useful for the discussion to follow to call a potential self-exciting system any nonself-exciting system which becomes self-exciting if the regenerative feedback is sufficiently increased or if the value of any other parameter is suitably modified.

We referred (section 5) to the phenomenon of forced or automatic synchronization which occurs in self-exciting systems subjected to the effect of a sinusoidal force which has a frequency close to their own frequency. Koga [27] as well as van der Pol and van der Mark [26] have observed similar phenomena in self-exciting systems subjected to the effect of a force whose frequency is a multiple of their own frequency. As for potential self-exciting systems, it is observed in this case that when the frequency w of the applied electromotive force is equal or approximately equal to a multiple of its own frequency, the mode of operation being suitably selected, there is a special phenomenon of synchronous excitation. As long as its own period is not close to a multiple of that of the applied force, a potential self-exciting system is the source of very weak "forced" oscillations. Nevertheless, when the system has its own frequency sufficiently close to ; $\frac{w}{n}$, n being an integer, there can appear intense oscillations of frequency exactly equal to $\frac{w_1}{n}$. This is the phenomenon of resonance of the n-th class [15], [37].

There still exists, in potential self-exciting systems, another phenomenon which should be mentioned here. The resonance of the n-th class requires a well-defined mode of operation of the tube. If, starting from this mode of operation, the regeneration is increased, quite slightly in order that the system does not become self-exciting, there may be seen to appear, under certain conditions, no matter what may be the period of the emf, intense oscillations almost identical to the own oscillations of the system. To these intense oscillations are added very weak "forced" oscillations in such a way that the phenomenon in its own system is almost periodic. It can be then termed asynchronous excitation [39], [40], [41].

In the more simple case of n = 2, when the emf $E = E_o \sin wt$ acts upon an oscillating circuit interposed between the plate and the filament of a tube whose characteristic can be represented by a polynomial of the third degree, the equation of the phenomenon (the grid current being assumed nonexistent) can be described in the form: /108

$$CL\frac{d^2 i}{dt^2} + CR\frac{di}{dt} + i = f_0\left(\frac{di}{dt}\right) + CE_0\,\omega\cos\omega t \tag{14}$$

in which

$$f_0\left(\frac{di}{dt}\right) = i_{a_0} + \alpha_0\frac{di}{dt} + \beta_0\left(\frac{di}{dt}\right)^2 + \gamma_0\left(\frac{di}{dt}\right)^3.$$

We assume that $\gamma_o < 0$. Using suitable transformations and notations:

[1] The capability for exciting a potential self-exciting system on a frequency equal to the half of that of the emf has been likewise reported by Groszkowski [38].

$$\tau = \frac{\omega t}{2}, \quad 2\vartheta = \frac{2R}{\omega L}, \quad \xi = \frac{\omega^2 - 4\omega_0^2}{4\omega_0^2}, \tag{15}$$

$$q = \frac{4M}{4} \frac{E_0}{v_0}, \quad x = M \frac{di}{dt} \quad (v_0 - \text{ saturation voltage.}$$

The equation can be rephrased in the form:

$$\ddot{x} + x = \mu f(x, \dot{x}) - 3q \sin 2\tau \tag{16}$$

in which

$$\mu f(x, \dot{x}) = \frac{d}{d\tau} \left[\frac{1}{1+\xi} f_1(v) - 2\vartheta x \right] +$$

$$+ \frac{\xi}{1+\xi} x = \mu \left[(k + 2x + \gamma_1 x^2) \dot{x} + \frac{\xi}{\beta} x \right];$$

$$\gamma_1 = \frac{3\gamma}{\beta} < 0, \quad \mu = \frac{\beta}{1+\xi} \tag{17}$$

and the regenerative factor

$$k = \frac{\alpha - 2\vartheta(1+\xi)}{\beta}$$

Applying to equation (16) the van der Pol method, we produce by the transformation:

$$\begin{aligned} x &= u \sin\tau - v \cos\tau + q \sin 2\tau; \\ \dot{x} &= u \cos\tau + v \sin\tau + 2q \cos 2\tau; \end{aligned} \tag{18}$$

the system of truncated equations:

$$\dot{u} = \frac{\mu}{2} \left\{ \left[k + \frac{\gamma_1}{4} (z + 2q^2) \right] u - v \left(q + \frac{\xi}{\beta} \right) \right\}; \tag{19}$$

$$\dot{v} = \frac{\mu}{2} \left\{ \left[k + \frac{\gamma_1}{4} (z + 2q^2) \right] v - u \left(q - \frac{\xi}{\beta} \right) \right\}; \quad (20)$$ /109

$z = u^2 + v^2$ being the square of the instantaneous amplitude.

These equations may easily be solved if $\psi = \frac{u}{v}$ and z [42] are selected as variables. Their initial values will be designated by ψ_0 and z_0. Granting that:

$$m = \sqrt{\frac{q + \frac{\xi}{\beta}}{q - \frac{\xi}{\beta}}}, \quad 2p = \mu \sqrt{q^2 - \frac{\xi^2}{\beta^2}}, \quad M = \mu \left(k + \frac{\gamma_1 q^2}{2} \right)$$

$$\sigma = \frac{\xi}{pq}, \quad z_{st} = \frac{4(M + 2p)}{\mu |\gamma_1|}; \quad (21)$$

we obtain the solution in the following form:

$$\psi = m \frac{m + \psi_0 - (m - \psi_0) e^{-2p\tau}}{m + \psi_0 + (m - \psi_0) e^{-2p\tau}}; \quad (22)$$

$$z = z_{st} \frac{C^2 - 2\sigma C e^{-2p\tau} + e^{-4p\tau}}{\left[\frac{z_{st}}{z_0} (C^2 - 2\sigma C + 1) - C^2 - 2\sigma \frac{M + 2p}{M} C + \frac{M + 2p}{M - 2p} \right] e^{-(M + 2p)\tau} + \ldots} \rightarrow$$

$$\rightarrow \ldots \frac{C^2 - 2\sigma C e^{-2p\tau} + e^{-4p\tau}}{+ \left[C^2 - 2\sigma \frac{M + 2p}{M} C e^{-2p\tau} + \frac{M + 2p}{M - 2p} e^{-4p\tau} \right]} \quad (23)$$

Equations (22) and (23) approximately describe what occurs beginning from any initial conditions whatsoever as well as the case when the system is not excited ($k < 0$) as well as when it is self-exciting ($k > 0$). If $q = 0, \varepsilon = 0$, $p = 0$, i.e. if there is no applied force, we resort to the van der Pol solution for autonomous systems [6].

It is evident from (23) that z only tends toward a constant value z_{st} different from zero when p is real and $M + 2p > 0$, i.e. when

$$q^2 > \frac{\xi^2}{\beta^2} \qquad (24)$$

and

$$k + \frac{\gamma_1 q^2}{2} + \sqrt{q^2 - \frac{\xi^2}{\beta^2}} > 0; \qquad (25)$$

these are the conditions of existence of constant solutions in the case of u and v. These conditions fulfilled, periodic oscillations (cf. formula (18) whose period is double that of the emf will establish themselves in the system. These oscillations with double period only appear in a limited frequency interval characteristic of the system and the phenomenon recalls, by its appearance, certain resonance phenomena. The term "resonance of the second class" (and, more generally, of the n-th class) recalls that this theory is closely connected to the existence of periodic solutions of the second class of Poincare [2]. /110

Being given that $\gamma_1 < 0$, condition (25) cannot be fulfilled in the case of a potential self-exciting system ($k < 0$) except when the value of q is included within a certain interval $q_{min} < q < q_{max}$. We state that there is a "threshold" and a "ceiling" for the value of emf which is capable of exciting in a potential self-exciting system double period oscillations. In the case of self-exciting systems, condition (25) is fulfilled no matter how small q may be. There is therefore no threshold for automatic synchronization of a self-exciting system with a period the double of that of the applied emf.

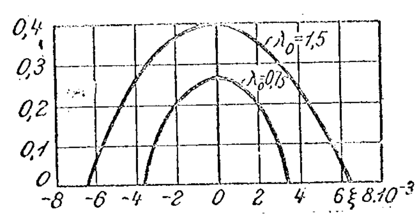

Figure 39. Resonance Curves of the Second (Theoretical) Class

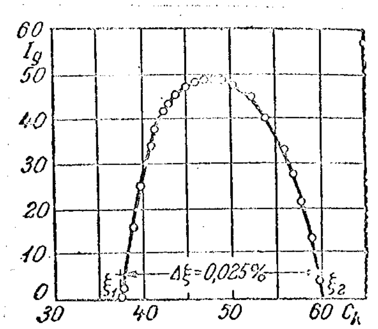

Figure 40. Resonance Curve of the Second (Experimental) Class. $w = 13.9 \cdot 10^5$

The square of the amplitude of the stationary oscillations with a period double of that of the emf which have been excited by resonance of the second class is provided by the formula:

$$z_{st} = \frac{4}{|\gamma_1|} \left[k + \frac{\gamma_1 q^2}{2} + \sqrt{q^2 - \frac{\xi^2}{\beta^2}} \right]. \qquad (26)$$

According to this formula, the stationary amplitude is the function of the detuning ε quite otherwise than in the case of ordinary resonance. The curves which provide z as a function of ε -- they can be termed resonance curves of the second class -- are shown in Figure 39 (theoretical) and in Figure 40 (experimental). Formula (26) likewise provides the stationary amplitude as a function of the value of the applied emf. This function ("characteristic of amplitude") is depicted in Figure 41 (theoretical) and in Figure 42 (experimental). Note that the "excitation band" (i.e. the frequency interval in which the second-class resonance occurs) equal to zero in case of $q = q_{min}$ becomes wider at first as q increases, then decreases and again drops to zero in the case of $q = q_{max}$. /111

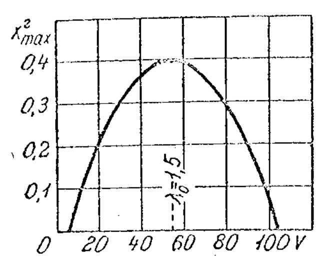

Figure 41. Characteristic of Amplitude (Theoretical)

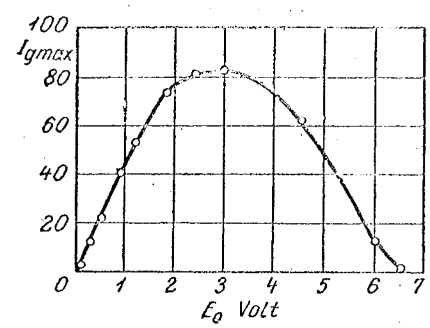

Figure 42. Characteristic of Amplitude (Experimental)

The increase of oscillations up to the stationary amplitude has quite another characteristic than in the case of ordinary resonance as shown in Figure 43. The latter depicts the variation of amplitude as a function of time beginning from the instant at which external force becomes a factor in the resonance of second class (curve 2) and in ordinary resonance (curve 1).

Let us note the similarity between curve 1 and the curve expressing the increase in amplitude of self-exciting phenomena. This similarity is not accidental. Whereas in ordinary resonance, the excitation of oscillations takes place no matter what the initial conditions may be and can start more

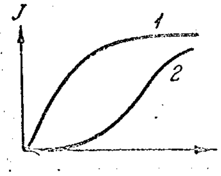

Figure 43

particularly beginning from absolute equilibrium ($i = 0$, $\frac{di}{dt} = 0$). In resonance of the second class, as in self-excitations of pulses, whether they are very small or not, they are necessary to cause the system to deviate from the initial state $z_0 = 0$. The position of equilibrium of a potential self-exciting system which satisfies resonance conditions of the second class becomes, under the effect of external force, an unstable focus enclosed in a stable limiting cycle.

The special characteristic of the curve of growth of the oscillations in resonance of the second class can be used advantageously for practical ends (cf. below).

When the system is in an abrupt mode of operation, resonance of the second class shows certain special features. Thus, there may be observed at the boundaries of the regions of excitation "resistance" phenomena owing to the partial superposition of different ranges of dynamic stability. If the characteristic of the tube is expressed by a fifth-degree polynomial, as has been done successfully by Appleton and van der Pol [43] as well as by other authors [44], in the case of self-excitations it is possible, by applying the methods of Poincare (cf. section 2), to provide an approximate theory which takes into account satisfactorily all the characteristic properties of the phenomena. Figure 44 provides the resonance curves of the abrupt mode of operation calculated in this manner. They agree closely with experimental results (Figure 45). /112

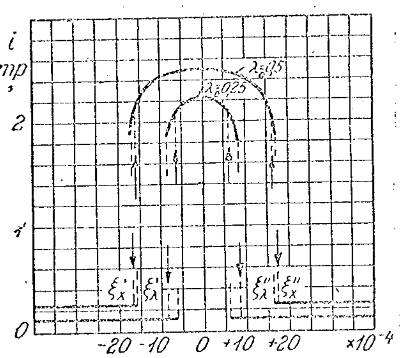

Figure 44

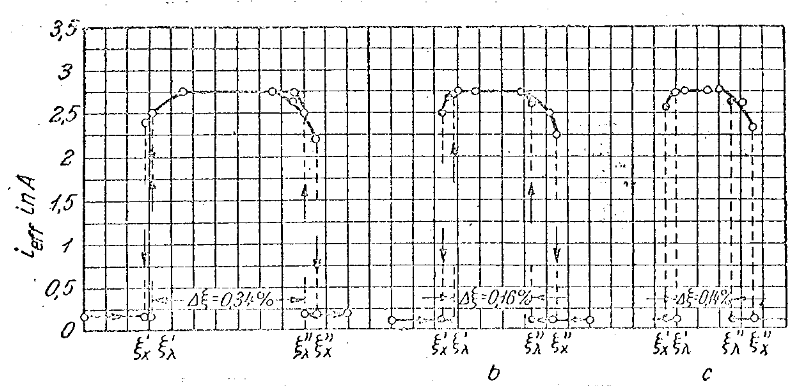

Figure 45. = 163 v_{eff}; b $E_0 = 101$ v_{eff}; c $E_0 = 79$ v_{eff}. v, $V_f = 8{,}34$ v, in the case M 28, $\omega = 1{,}047 \cdot 10^6$ of the tube

When an experimental study is made of the resonance of the second class in systems with abrupt excitation, certain precautions should be taken in order to avoid the phenomena of asynchronous excitation as defined above. It has been shown theoretically [39] and confirmed experimentally by E. Roubtchinski [45] in one of our laboratories (Central Radio Laboratory) that asynchronous excitation is only possible if the mode of operation is abrupt and if the values of the regenerative feedback and the amplitude of the emf are each included within a specific range. Figure 46 shows the modes of operation corresponding to the different values of the feedback factor k. The region to the right of zero is that of spontaneous excitation. Between zero and A ($0 > k > -\frac{\gamma_1^2}{8}|\varepsilon_1|$) is found the region of resistance. It is possible to produce, to the left of A, assuming the circuit is suitably tuned and that the emf has a suitable value, the resonance of the second class. In addition, in the shaded portion between A and B ($-\frac{\gamma_1^2}{8|\xi_1|} > k > -\frac{\gamma_1^2}{6|\xi_1|}$) there can occur phenomena of asynchronous excitation. In order to produce, in the pure state,
resonance of the second class, it is necessary to work to the left of B. This /113
is a very important matter for receiving stations making use of this phenomenon.

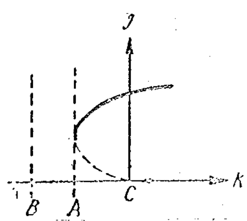

Figure 46

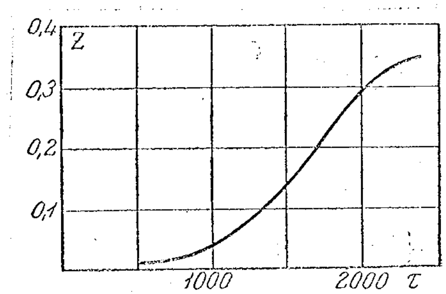

Figure 47

The transitory phenomena in resonance of the second class in an abrupt mode of operation have been studied by A. Melikian [46]. His experiments show that the phase is set up much more quickly than the amplitude. Theory provides, in the case of the soft mode of operation provided that customary numerical values of parameters are taken, the same result. Assuming this condition to be true as a hypothesis in his calculations, Melikian obtained for the abrupt mode of operation relatively simple theoretical formulae. Figure 47 shows one of the theoretical curves of growth of oscillations. We have reproduced on these pages some oscillograms produced by Melikian. The one shown in Figure 48 which was produced using an electronic oscillograph with an incandescent cathode requiring the synchronous repetition of the phenomenon shows the effect of a rectangularly shaped signal. The oscillograms shown in Figures 49 and 50 illustrate the increase and decrease of oscillations in a triode transmitter and in a potential self-exciting system

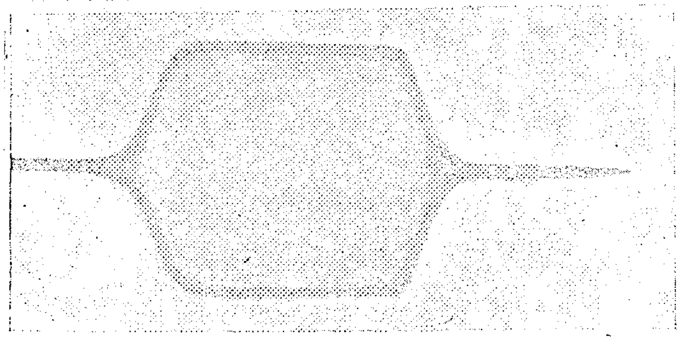

Figure 48

under the effect of a force of double frequency and of its own frequency. They were taken using an oscillograph with a cold cathode and with an inside photographic device which allowed the recording of a single event.

The theory developed for the resonance of the n-th class likewise allows the analysis of phenomena of automatic synchronization of a self-exciting system on a frequency equal to a submultiple of the frequency used [37]. Just as in the ordinary drive (cf. section 5) there is no threshold for the emf and the decrease of the latter only causes the synchronization band to become narrower. Theoretical and experimental data show that above a certain limit, the amplitude of the "self-excitations" decreases when the amplitude of the force increases and, beginning from a specific value of the latter, becomes equal to zero. The self-excitations are damped by the emf and there only remain in the system forced oscillations having the frequency of the emf. /114

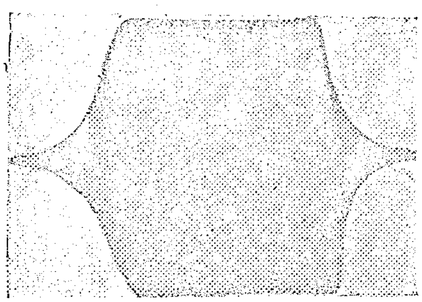

Figure 49

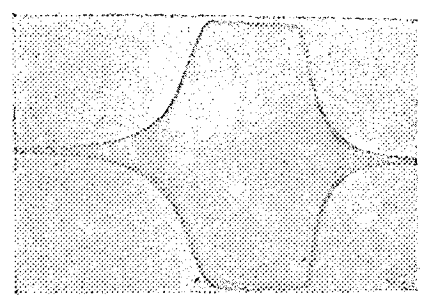

Figure 50

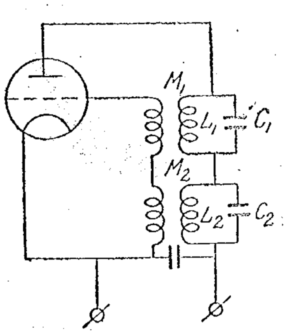

Figure 51

Theoretical investigations conducted by the Central Radio Laboratory have shown that it is impossible, in a potential self-exciting system with one degree of freedom and with a soft mode of operation, to produce oscillations whose resonance is of the third class. Experimental data show (Tschikhatchov) that oscillations synchronized in a period three times that of the emf or resulting from an asynchronous excitation can be "driven" to the left of B (Figure 46). In potential self-exciting systems with two degrees of freedom and more, it is possible to excite oscillations whose resonance is of a higher order. Tschikhatchov produced resonance phenomena of the fourth class in the installation described in Figure 51.

Resonance phenomena of the n-th class have a certain relationship with the excitation of oscillations by periodic variation of parameters of a system. This is the "parametric excitation" to be discussed in the following paragraph. In reality, we can consequently interpret, in a purely qualitative manner, the excitation by resonance of the n-th class. The external force acting on a potential self-oscillating system causes first of all to appear there "forced" oscillations which have the same period as the force. By reproducing the reasoning used to analyze the stability of a movement by the methods of Poincare and Liapounov, we can consider our system as nonlinear /115 and, in the vicinity of the forced oscillations, like a linear system whose parameters are functions of the "forced" solution q sin nt. The properties of this linear system with parameters varying periodically are well known (section 7). If it is located in one of the regions of instability, the forced oscillation will be unstable and the system will perform increasing oscillations whose frequency will be a submultiple of that of the force. This method of reasoning offers certain advantages, and it is often useful to consider the resonance of the n-th class as a sort of parametric excitation. In order to distinguish the parametric phenomena in their true sense which occur when the parameters of an electric or mechanical system are caused to vary indirectly with those of resonance of the n-th class, we shall call the first ones heteroparametric and the second ones self-parametric.

The phenomena which have just been described possess properties allowing hope for certain advantages in their practical application. Experience has shown that resonance of the second class can be used successfully for demultiplication of frequencies as well as the production of very high amplifications in cases in which the frequency should remain quite constant as, for example, in the case of emitters with independent excitation, and above all in the case of reception.

The resonance curves of the second class with abrupt edges, the existence of a threshold and a ceiling for applied emf give rise to new capabilities for selective reception. Nevertheless, it should naturally not be forgotten that with receivers as well as with high-speed operating automatic tranceivers, we are dealing not only with stationary phenomena, but with phenomena of increase and decrease which, as we have seen (Figure 43) are essentially different from those occurring in the case of phenomena with ordinary resonance.

Experiments performed in 1930 and 1931 under practical conditions have shown that the applications in a radio receiver of a device with resonance of the second class used as a selective filter ("self-parametric filter") gives excellent results. Figure 52 reproduces a photograph of two simultaneous recordings of signals emitted by radio station WCI (wavelength 16,317 meters) made on 4 February 1931 at the radio receiving exchange of Boutovo, near Moscow. Track a comes from a radio receiver furnished with a self-parametric

filter. Track b comes from a radio receiver which has a crystal filter. The /116
difference is obvious.

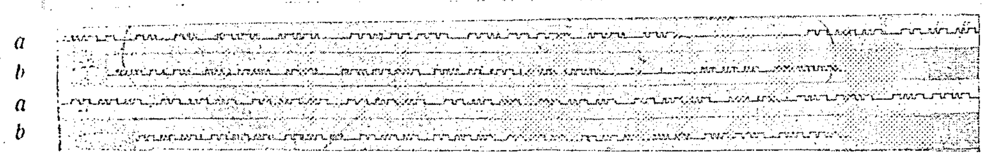

Figure 52

The prolonged tests of a radio receiver with self-parametric filter performed at Sagaredjo, near Tiflis, in a region which is subject to intense atmosphericdisturbances, have shown that this filter is very effective in separating a prolonged harmonic action from a brief pulse.

This property of systems with resonance of the second class is explained by the characteristic of the law according to which oscillations increase. Whereas, in a linear filter, a brief pulse (with respect to the duration of the point) which is sufficiently strong can cause oscillations whose amplitude is comparable or even greater than the one supplying the signal. Here, owing to the peculiar feature of the curve of increase, it only gives small oscillations. In this way, the self-parametric filter practically suppresses atmospheric disturbances which have the form of short pulses, although they may not reach a very considerable value. This insensitivity towards atmospheric pulses continues when they are superimposed on the signal. Nevertheless, very strong atmospheric discharges given effect of "fractioning" of the signal (observed by I. Borouchko and N. Weissbein in 1931). In the recordings on tape there are certain signs (dots or dashes) which show discontinuities (1, 3 on Figure 53).

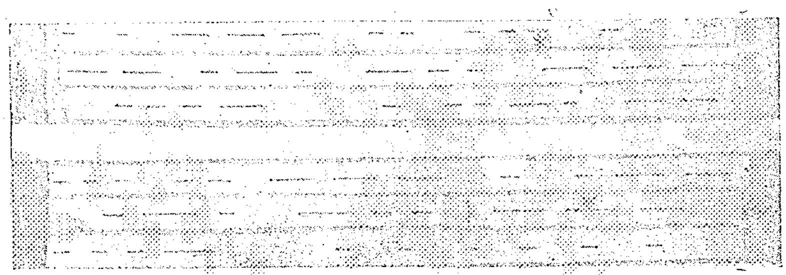

Figure 53

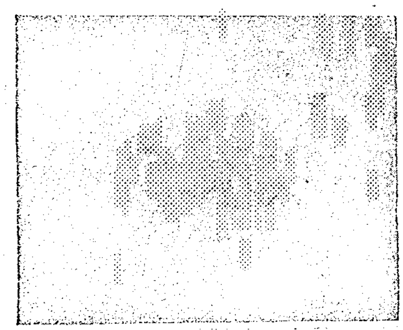

Figure 54

This last phenomenon was studied in one of our laboratories (the Central Radio Laboratory) by A. Melikian. Using the oscillographic method in a very skillful manner, he examined in detail the simultaneous action on a self-parametric system of a signal and a short pulse (series of damped oscillations). He showed that "fractioning" occurs when the pulse arrives at an instant in which the double period oscillations owing to the signal are already almost established (Figure 54). However, if the pulse arrives at a time when the double period oscillations are still weak, it accelerates their growth. Figures 55 and 56 allow comparison of the oscillograms of increase and decrease of a self-parametric system (Figure 55) with those of a linear system (Figure 56). /117

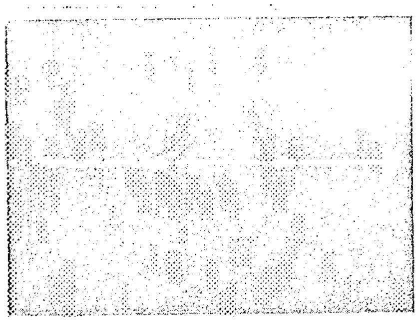

Figure 55

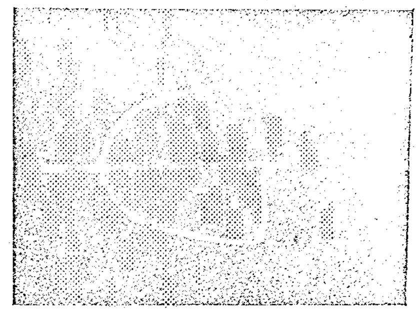

Figure 56

7. Parametric Excitation

The phenomena produced by an external action in nonlinear circuits (section 6) are closely related to the excitation of oscillations by periodic variation of the parameters of an oscillating system. This effect, that can be called for short parametric excitation, has been known for some time to physicists (Melde [47], Rayleigh [48]). The great importance that it has in radioengineering is likewise known. However, although the possibility of parametric excitation of electric oscillations has been known for a long time (Rayleigh [49], Poincare [50], Brillouin [51] and later van der Pol [52]), it is only in the last few years that this phenomenon was realized for its full value and its systematic study was undertaken. We should like to mention the experiments of Heegner [53] and Guenther Winter [54] on the excitement of electric oscillations of acoustic frequencies by alternately magnetizing the iron nucleus of a self-induction winding as well as the experiments of Guenther Winter [55] and I. Watanabe, T. Saito and K. Kaito [56] and the excitation of electric oscillations by mechanical periodic variation of the self-induction of an electric oscillating system.

We have likewise performed experiments on the parametric excitation of electric oscillations by mechanical periodic variation of the self-induction [57] of a circuit, but using very different devices from those used by Guenther Winter and Watanabe. In addition, we produced the parametric excitation of electric oscillations by periodic variation of the capacitance of a circuit [58][1].

The authors mentioned are limited, as to theory, to the use of the linear differential equation with periodical coefficients which provides the excitation conditions but can say nothing as to the capability and characteristic of a stationary mode of operation. Now, this question is no less important than the preceding one. This is why we begin with a general overview of the theory of parametric excitation, a theory which should be supported, in order to be complete, by a nonlinear differential equation. /118

It is easy to show by considerations of energy, that it is possible, by causing the capacitance of a circuit to suitably vary, to excite oscillations in the latter. Let us assume that at an initial stant t = 0, when the current is equal to zero and the condenser possesses a charge q, we reduce its capacitance by a small quantity ΔC. Having done this, we supply the work $\frac{\Delta C}{2C^2} q^2$. Let us then allow the condenser to discharge and at instant $t = \frac{T}{4}$ (T being the period of the circuit) when all the energy is magnetic and the charge of the condenser zero, let us restore its capacitance to its first value. We do this with no expenditure of work. at instant

$$t = \frac{T}{2}$$

the current is reduced to zero and the condenser carries a charge which is greater or less than q, according to which the energy supplied to the system as its capacitance is reduced represents an excess or deficiency with respect to the energy expended. At instant $t = \frac{T}{2}$, the cycle of variation of capacitance is complete. Let us phrase this otherwise. The oscillations will

---

[1] W. L. Barrow wrongly assumes that his experiments (*Proc. of the Inst. of Rad. Eng.*, Vol. 22, p. 210, 1934) shows the capability for the parametric excitation of an oscillating circuit by periodic variation of its capacitance. He causes the variation of not only the capacitance of an oscillating circuit, but the ohmic resistance of a shunt containing a condenser. Now, the variation of a positive resistance can be carried out (and is carried out) without expenditure of energy. This device therefore does not allow, by its principle alone, supplying to the circuit by mechanical work, of the energy necessary for the excitation and maintenance of oscillations. There is no doubt that Barrow has observed in his experiments not only the parametric excitation of oscillations by periodic variation of the capacitance, but also phenomena owing to the presence of an electronic tube and from regenerative feedback.

gradually increase no matter how small the initial load provided that the following condition be fulfilled:

$$\frac{\Delta C}{2C^2} q^2 > \frac{1}{2} R_i \frac{\pi}{\omega}, \quad m > \frac{\varepsilon}{2}$$

or

$$\varepsilon = \frac{\pi R}{L\omega},$$

ε being the mean logarithmic decrement of the system and

$$m = \frac{\Delta C}{2C} = \frac{C_{\max} - C_{\min}}{C_{\max} + C_{\min}}$$

being the "modulation rate" of the parameter.

The initial load q always is present, even in the absence of outside disturbed inductions (electric lines, atmospheric discharges), owing to statistical fluctuations.

We therefore can, by causing a periodic variation through a mechanical process of the capacitance of a circuit with a frequency the double of its own frequency and there excite electric oscillations without using any electromotive force. A similar reasoning is applicable in the case of variation of self-induction. /119

This abridged discussion is enough to show that in order to produce parametric excitation, two conditions should be satisfied:

1. The frequency of variation of the parameter should be suitably selected (in our example it is the double of the frequency belonging to the circuit).

2. In the case of a given mean logarithmic decrement, the modulation rate of the parameter should be sufficiently high[1].

---

[1] In case of sinusoidal variation of the capacitance, the condition $m > \frac{\varepsilon}{2}$ is to be replaced by $m > \frac{2}{\pi}\varepsilon$.

A more complete study of the initiation of oscillations in phenomena of parametric excitation, leads, as is known, to the discussion of "unstable" solutions of linear differential equations with periodic coefficients. If, for example, the capacitance varies according to the law:

$$\frac{1}{C} = \frac{1}{C_0} (1 + m \cos \nu t)$$

we have for

$$q = \int i \, dt$$

the equation

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} (1 + m \cos \nu t) \, q = 0 \tag{28}$$

which leads to the form

$$\ddot{x} + \lambda^2 (1 + m_1 \cos 2\tau) \, x = 0, \tag{29}$$

(equation of Mathieu), by granting

$$q = x e^{-\frac{R}{2L} t}, \quad \tau = \frac{\nu t}{2}, \quad \lambda^2 = \frac{2(\omega_0^2 - \delta^2)}{\nu^2}$$

$$\omega_0^2 = \frac{1}{LC_0}, \quad 2\delta = \frac{R}{L}, \quad m_1 = m \frac{\omega_0^2}{\omega_0^2 - \delta^2}.$$

Equation (29) was discussed from a mathematical viewpoint by Mathieu, Hill, Poincare, etc. It was also discussed, with respect to our present problem, by Rayleigh, then by Andronov and Leontowitsch [59] and by van der Pol and Strutt [60]. It is known that equations of the same type appear in a great number of problems of celestial mechanics, optics, elasticity, acoustics, etc.

The general solution of equation (29) is in the form:

$$x = C_1 e^{h\tau}\chi_0(\tau) + C_2 e^{-h\tau}\chi(-\tau),$$

$X(\delta)$ being a periodic function. In order for there to be parametric excitation, it is necessary for h to have a real part which has an absolute value which is greater than δ. This condition between the parameters λ and m defines the "unstable regions" of equation (28). They are located in the vicinity of the values $\frac{2w_1}{v} = n$, n being an integer. Their boundaries can be calculated by the approximative method of Rayleigh [61]. Thus, in the case of the first region of instability (n = 1), we have, with values on the order of approximately m^2: /120

$$\sqrt{1+\sqrt{\frac{m^2}{4}-4\vartheta^2}} \geq \frac{2\omega_1}{\nu} \geq \sqrt{1-\sqrt{\frac{m^2}{4}-4\vartheta^2}}. \qquad (30)$$

in order to find the second region (n = 2), terms on the order of m^4 should be taken into account. It follows that:

$$\sqrt{4+\frac{2}{3}m^2+\sqrt{m^4-64\vartheta^2}} \geq \frac{2\omega_1}{\nu} \geq$$
$$\geq \sqrt{4+\frac{2}{3}m^2-\sqrt{m^4-64\vartheta^2}}. \qquad (31)$$

The width of the regions of instability decreases as m^n.

As is shown by relations (4), (5) in order for initiation to be possible, it is necessary that, in the case of n = 1:

$$m > 4\vartheta \qquad (32)$$

and in the case of n = 2:

$$m > 2\sqrt{2\vartheta}. \qquad (33)$$

The modulation rate required for initiation is therefore greater, the decrement remaining the same, in the case of $n = 2$, than for $n = 1$. Initiation becomes still more difficult in the case of $n = 3, 4$, etc. This is why the case of $n = 1$ is, for practical purposes, the most interesting. It is the only one which we shall discuss here.

If the linear equation (29) was exact for any values of q, the amplitude of the oscillations, once conditions (30) or (31) were fulfilled, would increase beyond any limit. It is therefore necessary, in order for a system with periodically variable parameters to reach a stationary mode of operation and become a generator of alternating current, that it conform to a nonlinear differential equation[1]. In this case, the linear equation (2) is only valid (approximately) in the case of sufficiently small amplitudes. It allows only the setting up of conditions (30), (31) which should confirm the parameters for which there was initiation.

As will be seen, our experiments confirm this manner of regarding the phenomenon. In order to obtain a permanent mode of operation, it is necessary to introduce into the circuit nonlinear components such as an iron core coil, incandescent lamps, etc.; in the first case, the equation of the problem is: /121

$$\frac{d\varphi(\dot{q})}{dt} + R\dot{q} + \frac{1 + m\cos 2\omega t}{C_0} q = 0 \qquad (34)$$

with the nonlinear dependence ($\phi\mathring{q}$) of the flux with respect to the current being given in the form, for example, of a polynomial. The mathematical theory of the phenomena includes, in addition to the research of the condition in which the state of equilibrium becomes unstable (condition of initiation), the search for periodic solutions of equation (34) and discussion of their stability.

If the nonlinear part of the self is small with relation to its linear part and if, in addition, M is small, it is possible to apply to this equation the methods of section 3. In the most simple hypothesis in which:

$$\varphi(\dot{q}) = A + L_0\dot{q} + \beta\dot{q}^2 + \gamma\dot{q}^3 \qquad (35)$$

it follows by granting:

---

[1] The problem, likewise a nonlinear one, of the frequency modulation of a triode emitter has been studied by S. Rytov in an article carried in the journal of Techn. Phys. USSR.

$$\tau = \omega t, \quad \omega_0^2 = \frac{1}{L_0 C_0}, \quad \xi = \frac{\omega^2 - \omega_0^2}{\omega^2} \quad x = q/q_0$$

the equation:

$$\ddot{x} + x = \mu f(x, \dot{x}, \tau),$$

whose solutions approximately confirm the "truncated" equations:

$$\dot{u} = -\frac{1}{2}\left\{\left[\frac{m}{2}(1-\xi) + 2\vartheta\right] u + \left(\xi + \frac{\gamma_1}{4} z\right) v\right\},$$

$$\dot{v} = \frac{1}{2}\left\{\left[\frac{m}{2}(1-\xi) - 2\vartheta\right] v + \left(\xi + \frac{\gamma_1}{4} z\right) u\right\}$$

or which can be treated by the methods of section 2. The stationary solution is:

$$\frac{\gamma_1}{4} z = -\xi + \text{Sign}\,\gamma_1 \sqrt{\frac{m^2(1-\xi)^2}{4} - 4\delta^2} \tag{36}$$

whence the condition:

$$m(1-\xi) > 4\delta \tag{37}$$

practically identical to (32).

The curves of Figures (57) and (58), which can be called heteroparametric resonance curves (cf. section 5) provides z (the square of the amplitude) as a function of the "unbalance" ε. They differ essentially from the ordinary resonance curves and resonance curves of the second class.

As shown by Figure 57, whereas:

$$\xi < -\sqrt{\frac{m^2}{4}(1-\xi)^2 - 4\delta^2} \quad (\text{when} \quad \gamma_1 < 0)$$

no oscillation being observed. When: /122

$$\xi_1 = -\sqrt{\frac{m^2}{4}(1-\xi)^2 - 4\delta^2}$$

the oscillations begin, starting with small amplitudes and, z increasing, gradually amplifying. z increases linearly until, the unbalance assuming value

$$\xi_2^1 > \sqrt{\frac{m^2}{4}(1-\xi)^2 - 4\delta^2}$$

the oscillations stop abruptly.

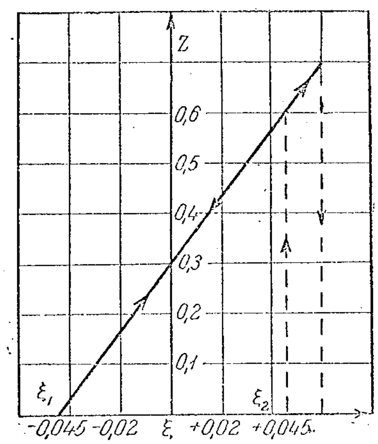

Figure 57 $\gamma_1 = -0{,}15$; $m = 0{,}15$; $\vartheta_0 = 0{,}03$.

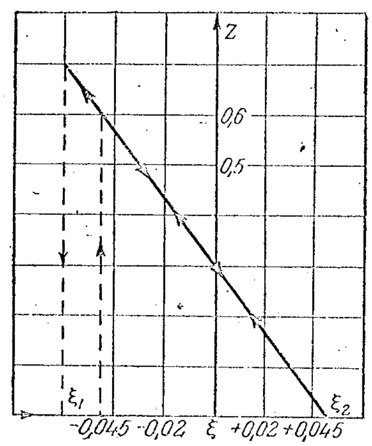

Figure 58 $\gamma_1 = +0{,}15$; $m = 0{,}15$; $\vartheta_0 = 0{,}03$.

As shown by formula (10), theory, at zero approximation, is only limited on one side, the unbalance ε. Stable finite amplitudes exist outside of the range of values of ε in which the conditions of initiation are fulfilled. In other words, the parametric oscillations, once excited, can be "driven" into the regions where the equilibrium is stable. When ε varies in a reverse direction, the oscillations appear when $\varepsilon = -\varepsilon_1$, then decrease in order to disappear when $\varepsilon = -\varepsilon_1$. Resistance therefore appears on only one side. In order to calculate the extent of the resistance circuit as well as to solve several other questions, μ will have to be used and the harmonics taken into consideration. Figure 58 shows that when $\gamma_1 > 0$ the phenomenon is reversed:

z increases whereas ε decreases, and the resistance circuit appears for positive values of $\varepsilon = \varepsilon_1$.

These two cases were observed experimentally by W. Lazarew [66]. V. Gouliaev and V. Migouline [62] have shown that the same results are produced by expressing the flux, as Dreyfus [63] and Zenneck [64] have proposed, by the function:

$$\Phi(\dot{q}) = \Phi_0 \arctan(k\dot{q}) + L_2\dot{q}.$$

We set as a goal in our experiments production of the effect of parametric excitation and the confirmation of the theory described above. /123

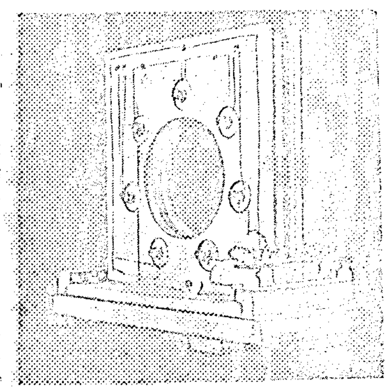

Figure 59

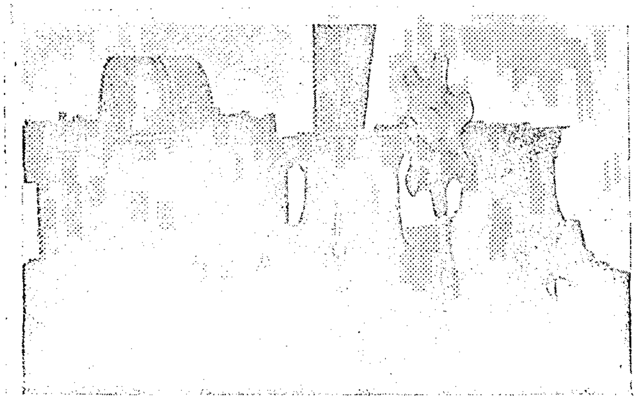

Figure 60

We shall cause to vary, in the first experiments, the self-induction of a circuit using a device that can be seen on Figures 59, 60, 61. The variable self-induction was composed of seven pairs of flat coils fastened face to face with two parallel disks on the periphery of two circumferences. A serrated metallic disk was able to rotate within the interstice contrived between the coils. The teeth, likewise seven in number, were cut in such a fashion as to fill, then empty simultaneously the field of the coils. As the disk spins, the self-induction of the circuit decreases when the teeth enter the field of the coils, then increases when they leave it. By using a disk made from duraluminum, we were able to achieve a peripheral velocity of 220 m/sec and, in this way, attain a considerable frequency of variation of the parameter (1700-2000 per/sec). The coils were supplied with iron cores divided in such a way as to increase self-induction and concentrate the field. This apparatus allowed production of parametric excitation of rather powerful oscillations in the circuit shown on Figure 62 which has no current or voltage source. By tuning this circuit on a frequency approximately equal to

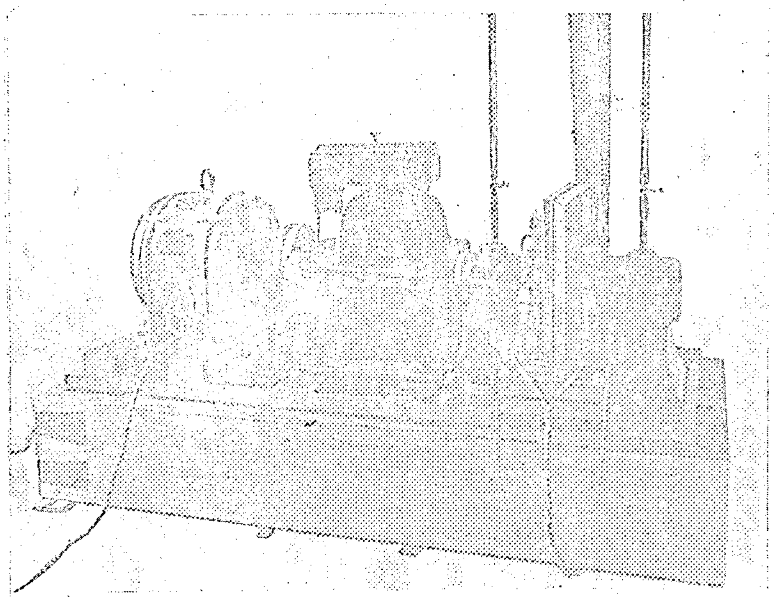

Figure 61

$\frac{w}{2}$, w being the frequency of variation of the self, there may be ascertained the presence of oscillations whose frequency is exactly equal to $\frac{w}{2}$. The amplitude is swiftly increased until the installation is broken either in the condenser or in the conductors of the circuit. In our experiments, the voltage went as high as 12,000-15,000 v. In order to produce a stationary mode of operation it was necessary, as required by theory, to introduce into the system a nonlinear component. This was, in the first experiments, a group of 100-watt incandescent lamps branched onto the oscillating circuit. /124

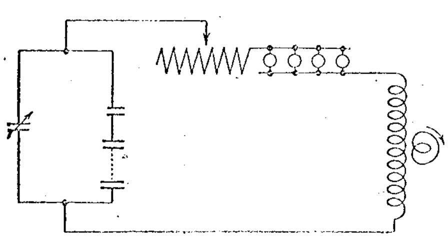

Figure 62

More detailed experiments were carried out in our laboratory of the Institute of Electrophysics of Leningrad by W. Lazarew [66] with an apparatus providing a greater modulation rate (40% instead of 14% in the first experiments) and a greater power (as high as 4 kw). The device causing the self-induction to vary is depicted in Figures 63 and 64. The duraluminum rotor had eight teeth. The self varied with the frequency of approximately 1900 per/sec, which provided oscillations of approximately 950 per/sec. Stationary mode of operation was produced using nonlinear self-induction either of iron cores of the coils of the stator, or of a coil with special iron core possessing an auxiliary winding for continuous current of magnetic induction. By causing the intensity of the latter to vary, it is possible to shift the operational point on the curve of magnetic induction of the iron and modify coefficients β and γ of formulae (35), (36).

By measuring the maximum damping occurring with initiation of oscillations, we find that experience agrees very satisfactorily with formula (4), as /125
shown by the table below. The experimental curves of "heteroparametric"

TABLE 1

	m	δ	$\frac{2}{n}\delta$
I	0,144	0,23	0,146
II	0,35	0,57	0,365
III	0,384	0,595	0,375

Tr. Note: Commas indicate decimal points.

resonance clearly have the features indicated below (Figures 65 and 66). They rise or fall according to whether the permanent mode of operation occurs owing to induction coils themselves or to a special coil magnetized by a continuous current. Figures 67 and 68 are oscillograms of the stationary current. Figure 69 is an oscillogram of the transitory mode of operation.

We used for the excitation of electric oscillations by mechanical periodic variation of capacitance a device whose circuit diagram is provided in Figure 70. The oscillating circuit is formed from condenser C with capacitance varying periodically, shunted from oil condenser C (serving to tune the circuit) and from the self-induction coil L (several sections of the secondary of a nonferrous inductor). Condenser C (Figure 71) includes two systems of armatures, one stationary (stator) and the other rotating (rotor). The stator is made up of 26 square aluminum plates, each one having 14 radial grooves ranged symmetrically. The rotor is an assembly of 25 circular aluminum plates which are perforated in the same manner as those of the stator and actuated by a continuous current motor with a maximum rate of 4000 rpm. When the motor performs at n revolutions per second, the capacitance varies with the frequency of 14 n per/sec. /126

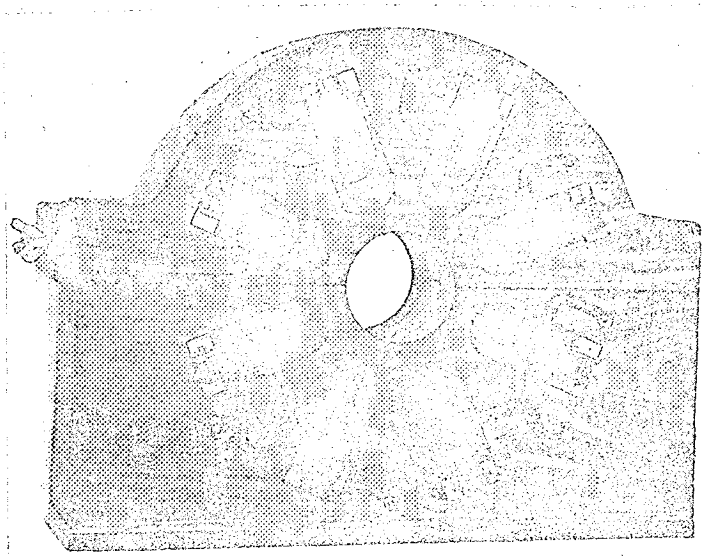

Figure 63

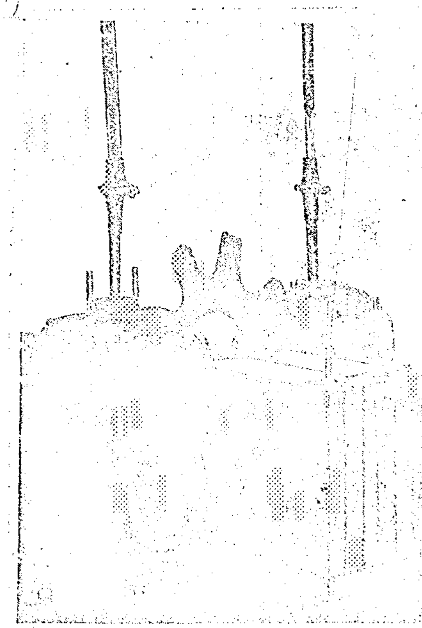

Figure 64

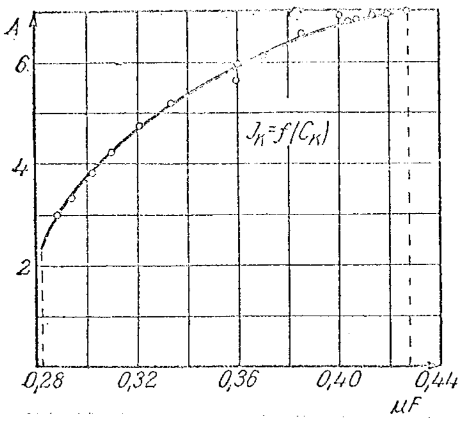

Figure 65

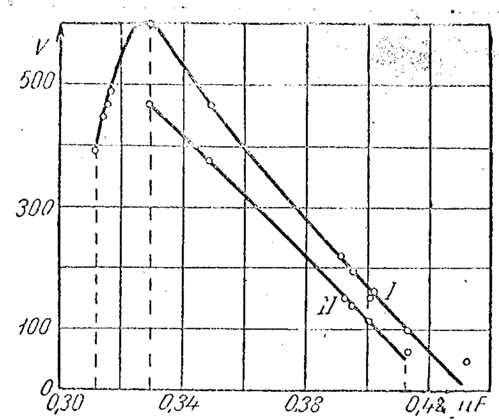

Figure 66

I. $v_k = f(C_k)$; $\vartheta_I = 0{,}414$; $i_d = 40$ mA
II. $v_k = f(C_k)$; $\vartheta_{II} = 0{,}47$.

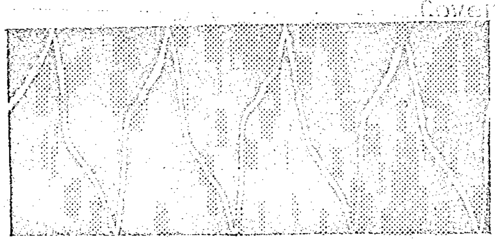

Figure 67

Figure 68

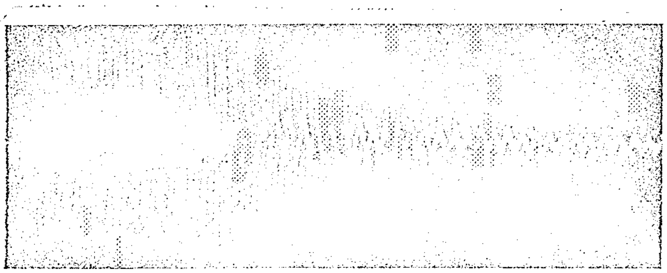

Figure 69

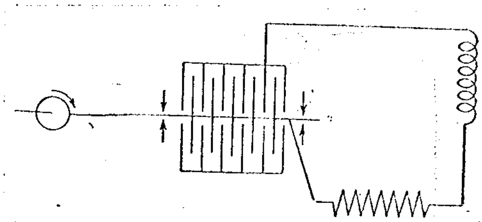

Figure 70

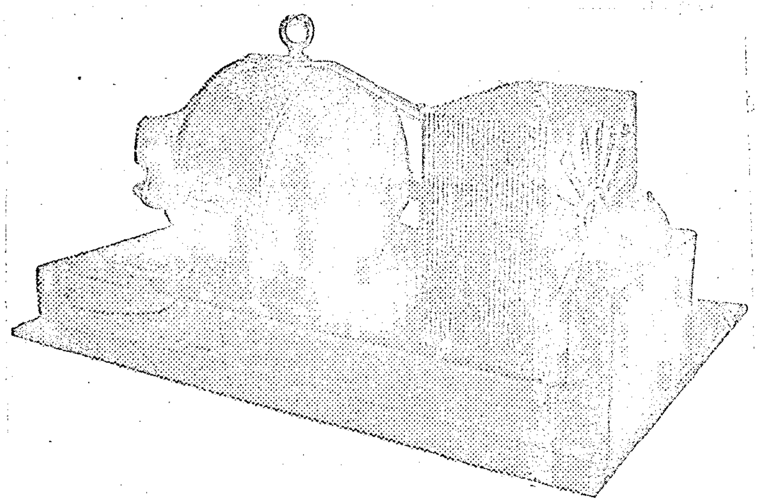

Figure 71

Six neon tubes with 220 v in series and a Hartmann-Braun static voltmeter of 1200 v allowed observation of the presence of oscillations and the evaluation of their intensity. Neon tubes were used at the same time to limit the increase in oscillations. /127

Since the rotor revolves at a fixed rate, there is a range of values of c in which the voltmeter fluctuates and the neon tubes light up. This range corresponds to the frequencies characteristic of the circuit in the vicinity of $\frac{n}{2}$. Controlling the frequency of the oscillations by means of a tuning fork, we were able to ascertain that it is constant within the whole range of excitation and equal to 7 n (n being measured with the tachometer).

If the neon tubes are removed, it is possible to predict, the system becoming linear, that the oscillations will increase until the installation is broken. This is what occurs in reality. The voltage, the neon tubes kept at 600-700 v, increases in their absence until a spark is produced between the armatures of the condenser (2000-3000 v). The frequency with which the spark occurs decreases proportionally as the frequency characteristic of the circuit draws apart from the value 7 n. This observation is likewise supported by theoretical considerations. According to the latter, the increase of oscillations is slowed down proportionally as the boundaries of the region of instability of the linear equation (29) is approached.

The modulation rate of the capacitance was, in these experiments, 0.175.

The experimental curves of Figure 72 provide the amplitude of the voltage excited as a function of the unbalance and the damping of the circuit. When the latter is increased, the range of parametric excitations is reduced. Its measured width is well in agreement with that provided by theoretical considerations.

In conclusion, let us add that H. Sekerska (Institute of Physics, Moscow) has provided the details of a new process allowing production of the Melde /128

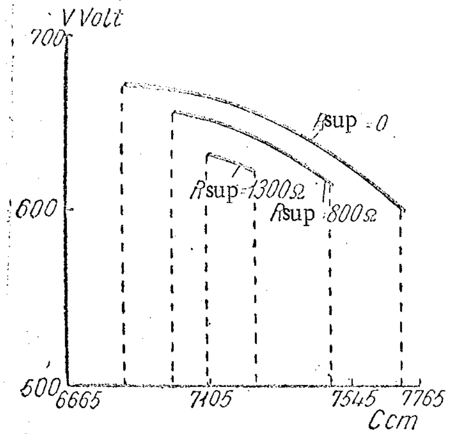

Figure 72

[47] phenomenon, i.e. the parametric excitation of a vibrating string[1]. A variable weight suspended by a metal wire allows tuning the normal modes of this latter on different frequencies. The wire closes the circuit of an alternating current of 50 per/sec. The temperature is caused to vary periodically with the frequency of 100 and, consequently, the voltage of the wire. When one of the normal modes of the wire is approximately tuned on frequency 50, there is observed, provided that the current intensity is sufficient, the parametric excitation of this mode.

8. Forced Oscillations of a System with Periodic Parameters. Parametric Coupling

If a system with periodically varying parameters is located within an unstable region, the stationary state cannot be described, as we have seen in section 7, by a linear equation. But if it is located in a stable region, the stationary mode differs from equilibrium only owing to the effect of a periodic or quasi-periodic external force. Since the oscillations are small enough, the phenomenon can then be described by an inhomogeneous linear equation with periodic coefficients which, in the most simple case, is in the form:

$$\ddot{q} + 2\delta\dot{q} + \rho(t)q = f(t) \tag{38}$$

in which δ is a coefficient of damping, $\rho(t)$ a periodic function and $f(t)$ a periodic or quasi-periodic function. The phenomena expressed by equation (38) depict a generalization of the well known phenomena of resonances produced by the action of a periodic or quasi-periodic force $f(t)$ on a linear system with constant parameters (harmonic resonator). These phenomena of generalized resonance were subjected by G. Gorelik[2] to a detailed theoretical study based on some considerations of principle as provided by one from our group [57].

The phenomena which have their source in harmonic resonators endow a special physical importance to the sinusoidal functions and harmonic analysis

[1] The publication is in process.
[2] The article in this issue of Techn. Phys. of the USSR.

of an arbitrary function. It is the language of sinusoidal functions which is used by theory to cope with resonance phenomena in the systems with constant parameters. However, this language ceases to be able to cope with systems with periodically varying parameters. Function f(t) should satisfy certain conditions in order for there to be resonance. These involve the laws of selection, the shape of forced oscillations which are expressed for the resonator with periodic parameters using new periodic or quasi-periodic functions. These can be considered as a generalization of the sine and cosine functions, and are determined by the intrinsic properties of the resonator. /129

Resonance phenomena show in linear systems with periodic parameters a character which changes depending on whether the resonator, in the absence of external force and damping, i.e. the ideal system described by the equation

$$\ddot{q} + p(t)q = 0$$

is in a stable region, at the boundary of an unstable region or in an unstable region. In the first case, the forced oscillations of resonance are proportional to $\frac{1}{\delta}$ just as in the case of the harmonic resonator. In the second case, there are two kinds of resonance: a "strong" resonance in which the forced oscillations are proportional to $\frac{1}{\delta^2}$ and a "weak" resonance in which they are proportional to $\frac{1}{\delta}$. If $f(t) = E \cos (wt + \phi)$, it is possible, by causing the ϕ phase to vary, make a transition from the weak resonance. In the third case, the resonance becomes more accentuated as the modulation rate increases. If the force is sinusoidal, the nature of the phenomena is likewise a function of its phase.

The theory of resonators with periodically varying parameters takes into account certain phenomena which have some similarity with those occurring in regenerative receivers. In the latter, the coupling between the grid circuits and the plate "regenerates" the circuit. It allows partial restoration, at the expense of the plate battery, of the energy dissipated in the forced oscillations. The theory of "regeneration" can be made, by disregarding the nonlinear terms of the tube characteristic, using a linear equation with constant coefficients. The regeneration decreases the coefficient of the dissipating term.

There is likewise produced a "regeneration" effect. In other words, it is possible to partially compensate, through utilization of a local source, for the losses of energy in a circuit performing forced oscillations, if one of its parameters is caused to vary at a suitable frequency. The phenomenon is especially advantageous if the frequency characteristic of the circuit, that of the emf and that of the variation of the parameter are in the ratio of 1:1:2. The modulation rate plays a role similar to that of the coupling coefficient in the regenerative receiver. One essential difference between the customary regeneration and this "parametric regeneration" means that the latter is essentially more a function of the phase than the emf with respect to the variation of the parameter. The "parametric regeneration" effect was observed and studied at the Central Radio Laboratory by Divilkovski and Rytov, as well as by Roubtchinski[1]. /130

The works of our laboratories have revealed new effects of parametric coupling between oscillating systems. They differ essentially from the well known phenomena which take place in linear coupled systems.

Let us take, for example, a mass suspended by a string with one fixed point. This is the elastic pendulum studied by G. Gorelik and one of our group [67] in relation to a question of optics[2]. When the mass oscillates vertically, the length of the pendulum undergoes a periodic variation. If the frequency of the elastic oscillations is double that of the angular oscillations, there will be parametric excitation of the latter by the former. (Whence the term "parametric coupling.") This phenomenon has, with respect to those phenomena discussed in section 7, this difference in that the variation of the parameter itself is a function of the oscillation which it excites. In reality, the angular oscillation causes the appearance of a centrifugal force of frequency equal to that of elastic oscillations and consequently reacts on the latter by ordinary resonance. The coupling is expressed, in the differential equations of the system, naturally autonomous, by nonlinear terms. Parametric coupling can likewise be seen in self-exciting systems with two degrees of freedom, for example that of Figure 51 in which it was studied by Tourbovitsch[3]. Since the operational point is selected in such a manner that the polynomial expressing the characteristic of the tube has a term of the second degree which is clearly marked, the equations of the system will be in the form:

$$\ddot{x} + \omega^2 x = 2\beta x\dot{y} + \cdots$$

$$\ddot{y} + 4\omega^2 y = \beta \dot{y}^2 + \cdots$$

(we only write the most significant terms). It can be seen that oscillation y of the frequency circuit 2w, causes variation at this critical frequency of "resistance" of the frequency circuit w (parametric action) and that, in

1 The publications are in preparation.
2 This model allows providing a standard qualitative table of certain anomalies of combinative diffusion (Raman effect) with CO_2 molecules which Fermi (*Zeitschr. fuer Phys.*, No. 71, p. 250, 1931) treated by suitable methods of quantitative mechanics.
3 Publication is in preparation.

return, oscillation x generates a frequency force 2w which reacts by resonance on oscillation y.

A. Tscharakhtschian[1] studied the action of a sinusoidal force on two circuits in parametric coupling forming a "parametric transformer." In this system, the variation of the current in the primary circuit causes the induction coil of the secondary circuit to vary by modifying the magnetization of the iron core coils. This allows production of phenomena of parametric excitation. /131

9. Role of Statistics in Dynamic Systems

We shall conclude by a few words on some questions whose theoretical and experimental study has just been begun by our laboratories[2] and which relate the theory of oscillations to statistical theories.

Even in the most simple case of the initiation of oscillations in a triode emitter, there can clearly be seen the rôle of statistics in the behavior of a system [69], [70]. Even without a regular deviation, the system, if at the initial instant, it is found in a state of equilibrium, it will always draw apart owing to random pulses [71] (produced, for example, by fluctuations). Now, the time during which the system arrives at a stationary state is a function of the value of the initial perturbation (this, of course, concerns the time necessary for the state of the system to arrive at a difference from the stationary state by one given value). However, in the triode emitter, the oscillating circuit levels the pulses to a "mean." This is why their influence is always shown by the formation of small oscillation of characteristic of a circuit whose amplitude is a function of the spectral intensity of the pulses.

This leveling of the pulses will not have the time to be carried out if the system is a very "high-speed" one. The instantaneous values of the current and voltage will then be random. In systems such that a small variation of initial conditions inside the establishment of such and such a final state, there will be observed directly the effect of various initial conditions. This is the case, for example, of the tip relays which have a saddle (at the origin of the coordinates) and two stable nodes, one on the right and the other on the left of the saddle. It is possible, for example, to produce a relay having two stable nodes located, on the phase plane, to the right and to the left of a saddle O, and symmetrical with respect to the latter. Let us assume that when the relay is triggered, the figurative point is located at O. It is clear that if the initial conditions are divided according to the laws of chance, the deviations to the right and to the left will be likewise probable and if, consequently, the system is triggered

[1] The publication is in preparation.
[2] The experiments are still in progress.

without applying the pulse from outside, it will travel either toward the right node or toward the left node, according to a statistical law. This statistical law will be modified if regular pulses are used. By comparing the effect of regular pulses with those of random pulses, it is possible to evaluate the value of the latter. Experiments of this kind were performed in one of our laboratories. The amplified fluctuations of the current of a vacuum tube were used as a source of random pulses. The value of the fluctuations determined in this way agrees satisfactorily with the well known theoretical and experimental findings of Schottky.

It is possible to raise another question concerning the transition of a system from one state to the other owing to the effect of random pulses. This problem was handled theoretically using the Fokker equation. In particular, L. Pontriaguine calculated the mathematical expectancy of the duration of transition from one state to the other. Using results obtained, it is possible to compute the duration of transition from one stationary state to another, thus allowing discussion of mean duration of stay in such and such a stationary state. This is naturally a function of the value of the random pulses. We were able to experimentally ascertain the existence of these "spontaneous" transitions from one stationary state to the other. /132

Observing the duration of stay in such and such a stationary state, it is possible to determine by using some plausible supplementary postulates, the value of random pulses. Note that the existence of random pulses limits in principle the precision with which it is possible to attribute to an oscillatory phenomenon a determined period.

Conclusion of the Oscillations

We desired to provide a short overview of some works carried out during the last few years in the laboratory of the Institute of Physics of the University of Moscow, the Central Radio Laboratory (Leningrad), the Laboratory of Nonlinear Oscillations of the Institute of Electrophysics (Leningrad) and the University of Gorki. In order not to overload our report, we omitted a whole series of questions relating, for example, to systems with several degrees of freedom[1]. And in the ones which we have discussed, we have had to confine ourselves to the most essential topics, in this way sacrificing a great number of often interesting details. We have provided references below. Soon a certain number of works so far only published in Russian will likewise appear in other languages.

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[1] The theory of the two coupled emitters was discussed by Mayer. The one for the oscillations forced into the self-exciting systems with two degrees of freedom gave rise to the works of Rytov, Bernstein and Ikonnikov and Mayer.

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